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9/6

SOLUTIONS OF THE CAMBRIDGE

SENATE-HOUSE PROBLEMS

FOR FOUR YEARS:—1848-51.

BY

N. M. FERRERS, B.A.,

OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE,

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1851.

LOAN STACK

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P R E F A C E.

It will, we believe, be universally admitted that there is no easier means of becoming acquainted with any branch of Mathematics, than the study of Examples illustrative of its principles. It is also indispensably necessary that the ingenuity of the Student be thoroughly exercised in attempting to discover for himself the solution of any problem which may be put before him: it is by no means our object, in publishing this book, to save him the trouble of doing so. But we believe that if, after having done his best to master a problem for himself, he is still unsuccessful, he will then derive great benefit from referring to the solution obtained by another person. It is for this purpose that we hope the present collection will be found of service.

Of the intrinsic value of these Problems we could have no doubt, even if we knew less of them, coming as they do from such high authorities. We have as little doubt that they are on all accounts the best problems that we could have chosen for solution for the benefit of the Student, for their general value, their variety, and because they shew what Senate-house Problems are, and are likely to be in coming years.

Part I. contains the solutions of those proposed in the first three days of examination: they are of a simpler kind than those proposed in the remaining five days, the solutions of which form Part II. The solutions of many problems have been kindly furnished by the Moderators by whom they were proposed: we take this opportunity of returning our acknowledgments to them and others of our friends who have assisted us in the progress of the work. We shall also feel much obliged to any of our readers who will send us either corrections of our solutions or improvements upon them.

Some difficulty, it will easily be understood, has been found in bringing all the problems to appear in their right places: any problems, however, which have been omitted in the body of the work, will be found in the Appendix. The problems on pp. 226, 241, should have been placed among the Trigonometry of 1850, and the Geometry of Three Dimensions of 1851, respectively.

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PART I.

SOLUTIONS OF THE SENATE-HOUSE PROBLEMS.

EUCLID.

ERRATA.

PAGE	LINE
87,	5 from bottom, <i>for</i> p' <i>read</i> n' .
95,	16, <i>for</i> $a_{p-r+1}a_p$ <i>read</i> $a_{p-r+1} + a_p$.
—	17, <i>for</i> $\dots + a_{p-r}) + a_{p-1}$ <i>read</i> $\dots + a_{p-1}) \quad a_p$.
176,	12, <i>for</i> $h^2 - 2ah$ <i>read</i> $h^2 - 2ah$.
288,	8, <i>for</i> R' <i>read</i> R_y .
215,	1, supply reference to figure (88).
231,	1, <i>for</i> (fig. 98) <i>read</i> (fig. 89).
Figure 46, <i>for</i> $p \ y \ z$ <i>read</i> $z \ p \ y$.	

Again, because EF is a diameter of the circle, therefore the angle FCE is a right angle. But CE bisects the right angle ACB, therefore ACE is half a right angle, therefore also FCA is half a right angle; that is, FC bisects the supplement of the right angle ACB.

2. A, B, C, (fig. 2) are three given points in the circumference of a circle; find a point P, such that if AP, BP, CP meet the circumference in D, E, F, the arcs DE, EF, may be equal to given arcs.

Join AB, and on it describe a segment of a circle, containing an angle equal to the sum of those subtended at the circum-

SOLUTIONS OF THE SENATE-HOUSE PROBLEMS.

EUCLID.

1848.

1. If the hypotenuse AB (fig. 1) of a right-angled triangle ABC be bisected in D , and EDF drawn perpendicular to AB , and DE , DF cut off each equal to DA , and CE , CF joined; prove that the last two lines will bisect the angle at C and its supplement respectively.

Join CD , then shall CD be equal to half the hypotenuse AB , that is, to DE or DF ; therefore a circle described from centre D , with radius DC , will pass through B , E , A , F . Let this circle be described, then the angle ECB , at the circumference, is equal to half the angle EDB at the centre, that is to half a right angle, and therefore to half the angle ACB ; that is, CE bisects the angle ACB .

Again, because EF is a diameter of the circle, therefore the angle FCE is a right angle. But CE bisects the right angle ACB , therefore ACE is half a right angle, therefore also FCA is half a right angle; that is, FC bisects the supplement of the right angle ACB .

2. A , B , C , (fig. 2) are three given points in the circumference of a circle; find a point P , such that if AP , BP , CP meet the circumference in D , E , F , the arcs DE , EF , may be equal to given arcs.

Join AB , and on it describe a segment of a circle, containing an angle equal to the sum of those subtended at the circum-

ference by the arcs AB and DE . Also join BC , and on it describe a segment of a circle containing an angle equal to the sum of those subtended at the circumference by the arcs BC and EF . These segments shall intersect in the required point P .

For join AP , BP , CP , and produce them to meet the circumference in D , E , F , respectively. Join AE , then the angle EAD is equal to the difference of the angles APB , APF , that is, to the angle required to be subtended by the arc DE . Therefore DE is equal to the arc required. Similarly it may be shewn that EF is equal to the arc required, and therefore P is the required point.

1849.

1. Through a point C (fig. 3) in the circumference of a circle, two straight lines ACB , DCE , are drawn, cutting the circle in B and E ; prove that the straight line which bisects the angles ACE , DCB , meets the circle in a point equidistant from B and E .

Let CP be the line bisecting the angles ACE , BCD ; P the point in which it meets the circle. Join PB , PE , BE ; then because the angles PBE , PCE are in the same segment, therefore they are equal to one another.

Again, because the angles BCP , BEP are opposite angles of a quadrilateral inscribed in a circle, therefore they are together equal to two right angles, that is to ACP and BCP . Therefore, taking away the common angle BCP , ACP is equal to BEP . But ACP is equal to ECP by construction, therefore from above ACP is equal to EBP : and it has been shewn to be equal to BEP , therefore the angles EBP , BEP are equal to one another, therefore PE is equal to PB . That is, the point P in which the bisecting line CP meets the circle is equidistant from B and E .

2. Two circles intersect in A and B (fig. 4). At A , the tangents AC , AD are drawn to each circle and terminated by the circumference of the other. If BC , BD be joined, shew that AB , or AB produced if necessary, bisects the angle CBD .

Produce CA, DA, to E, F. Then the angle CAF is equal to the angle DAE: but the angle CAF is equal to the angle ABC in the alternate segment, also the angle DAE is equal to the angle ABD in the alternate segment. Therefore the angles ABC, ABD are equal to one another, and AB, produced if necessary, bisects the angle CBD.

3. Draw a line to touch one given circle, so that the part of it contained by another given circle may be equal to a given straight line, not greater than the diameter of this latter circle.

Let ABC, DEF (fig. 5) be two given circles; it is required to draw a straight line touching the circle ABC, so that the part of it contained by DEF may be equal to a given straight line, not greater than the diameter of DEF.

In the circle DEF place the straight line DE, equal to the given straight line. Find G the centre of this circle, and with G as centre describe a circle touching DE. Draw AFH a common tangent to this latter circle and ABC, cutting DEF in F, H, this shall be the line required.

For since FH, DE each touch a circle whose centre is G, therefore they are equidistant from G, the centre of the circle DEF. Therefore FH is equal to DE.

Hence AFH is drawn touching the circle ABC, and the part of it contained by DEF is equal to the given straight line.

4. A quadrilateral figure possesses the following property: any point being taken, and four triangles formed by joining this point with the angular points of the figure, the centres of gravity of these triangles lie in the circumference of a circle; prove that the diagonals of this quadrilateral are at right angles to each other.

Let ABCD (fig. 6) be the quadrilateral, P any point within it: E, F, H, K, the middle points of the sides. Join PE, PF, PH, PK. And in them take PG_1 , PG_2 , PG_3 , PG_4 , respectively equal to two-thirds of PE, PF, PH, PK. $G_1G_2G_3G_4$ shall be the centres of gravity of the triangles PAB, PBC, PCD, PDA. Join EF, FH, HK, KE, these lines are evidently parallel to

$G_1G_2, G_2G_3, G_3G_4, G_4G_1$; and therefore E, F, H, K , lie in the circumference of the same circle. But EF, HK are each parallel to the diagonal AC , therefore also to each other. Similarly FH, EK are parallel to each other, therefore $EFHK$ is a parallelogram. And since it is inscribable in a circle, each of its angles is a right angle. Therefore also the diagonals AC, BD , which are respectively parallel to the sides of the parallelogram, are at right angles to each other.

1850.

1. If $ABCD$ (fig. 7) be a parallelogram, and P, Q two points in a line parallel to AB , and if PA, QB meet in R , and PD, QC in S , prove that RS is parallel to AD .

Because CD is parallel to QP , therefore SD is to SP as CD to PQ . And because AB is parallel to PQ , therefore RA is to RP as AB to PQ . But AB is equal to CD , therefore RA is to RP as SD to SP , therefore RP is to AP as SP to DP , therefore RS is parallel to AD .

2. Two sides of a triangle, whose perimeter is constant, are given in position; shew that the third side always touches a certain circle.

Let ABC (fig. 8) represent the triangle; AB, AC being the sides given in position. Describe a circle DEF touching BC and AB, AC produced. The side BC shall always touch the circle DEF .

For since BD, BF both touch the same circle, therefore BD is equal to BF . Hence AD is equal to AB, BF together.

Similarly AE is equal to AC, CF together.

Therefore AD, AE together are equal to AB, AC, BC together; that is, to the perimeter of the triangle ABC , which is constant. But since AD, AE touch the same circle, therefore AD is equal to AE , and their sum has been shewn to be constant; therefore AD, AE are each constant, that is, the circle touching BC , and AB, AC produced, touches AB, AC in fixed points; that is, it is a fixed circle. Therefore BC always touches a fixed circle.

1851.

1. In AB, the diameter of a circle, take two points C, D, equally distant from the centre, and from any point E in the circumference draw EC, ED; shew that

$$EC^2 + ED^2 = AC^2 + AD^2.$$

Take O (fig. 9) the centre of the circle, and join EO, and draw EF perpendicular to AB.

Then because CD is bisected in O, and produced to A,

$$\therefore AC^2 + AD^2 = 2(AO^2 + OC^2) \text{ (} \textit{Euc. II. 10} \text{)}.$$

Again, because EC is opposite to an acute angle O of a triangle ECO, therefore

$$EC^2 + 2OC.OF = EO^2 + OC^2 \text{ (} \textit{Euc. II. 13} \text{)}.$$

And because ED is opposite to the obtuse angle O of a triangle EOD, therefore

$$\begin{aligned} ED^2 &= EO^2 + OD^2 + 2OD.OF \text{ (} \textit{Euc. II. 12} \text{)}, \\ &= EO^2 + OC^2 + 2OC.OF; \end{aligned}$$

$$\begin{aligned} \therefore EC^2 + ED^2 &= 2(EO^2 + OC^2), \\ &= 2(AO^2 + OC^2), \\ &= AC^2 + AD^2 \text{ from above.} \end{aligned}$$

2. If through the fixed points P, Q, (fig. 10) parallel lines be drawn meeting two fixed parallel lines in the points M, N; then the line through the points M, N, passes through a fixed point.

Join PQ, and let it meet MN in O, and the given pair of parallels in R, S, O shall be a fixed point.

For since QN is parallel to PM, therefore QO is to PQ as ON to NM. And since NR is parallel to MS, therefore OR is to RS as ON to NM. Therefore OR is to RS as OQ to QP, or OR is to OQ as RS to QP; that is, QR is divided in a constant ratio in O, and therefore O is a fixed point.

3. In a given circle it is required to inscribe a triangle, similar and similarly situated to a given triangle.

Let ABC (fig. 11) be a given triangle, DEF a given circle; it is required to inscribe in DEF a triangle, similar and similarly situated to ABC .

At the point A in the straight line AB , make the angle BAG equal to the angle ACB . Find H , the centre of the circle DEF , and draw HK parallel to AG , HD perpendicular to HK . Through D draw DE , DF parallel respectively to AB , AC , DEF shall be the triangle required.

For draw DL parallel to HK or AG , and therefore touching the circle at D . Then the angle LDE is equal to the angle GAB . But GAB is by construction equal to ACB , and LDE is equal to DFE in the alternate segment. Therefore the angle DFE is equal to ACB . Similarly the angle DEF is equal to the angle ABC . Therefore the remaining angle EDF is equal to BAC , so that the triangles ABC , DEF are similar. And since the sides DE , DF are parallel respectively to AB , AC , therefore EF is parallel to BC , and they are similarly situated.

4. Describe a circle which shall pass through two points, and cut off from a given straight line a chord of given length.

Let A , B (fig. 12) be two given points, CX a given line, it is required to describe a circle passing through A , B , and cutting off from CX a chord of given length.

Join BA , and produce it to meet CX in C . Bisect AB in E , and draw EF perpendicular to AB . With A as centre, and radius equal to half the required chord, describe a circle FGH , cutting EF in F . With F as centre, and FA as radius, describe a circle BAG . Join CF , and let it cut the circle BAG in K , L . From CX cut off CM , equal to CK . The circle described through A , B , M shall be the circle required; that is, if N be the second point in which it meets CX , MN shall be equal to the required chord.

For the rectangle $CM.CN$ is equal to $CA.CB$ by the property of the circle ABM . And the rectangle $CA.CB$ is equal to $CK.CL$ by the property of the circle KAB . Therefore the rectangle $CK.CL$ is equal to $CM.CN$.

Let ABC (fig. 15) denote the triangle in its first position. At the point A , in the straight line AB , make the angle BAC' equal to the angle BAC , and make AC' equal to AC . Again, make the angle $C'AB'$ equal to the angle BAC' , and AB' equal to AB . Join $B'C'$; then in the triangles $CAB, C'AB'$, the sides CA, AB are respectively equal to $C'A, AB'$, and the angle CAB is equal to the angle $C'AB'$; therefore the triangles are equal in all respects. And the angle $B'AB$ is double of the angle CAB , therefore $C'AB'$ is the position of the triangle after revolving round A through an angle equal to $2.CAB$.

Again, join BC' ; make the angle $BC'A'$ equal to the angle $B'C'A$ or BCA , and $C'A'$ equal to $C'A$. Join $A'B$; then in the triangles $AC'B', A'C'B$, the sides $AC', C'B'$ are respectively equal to $A'C', C'B$, and the angle $AC'B'$ is equal to $A'C'B$. Therefore the triangle $A'BC'$ is equal in all respects to $AB'C'$, and therefore to ABC . And the angles $BC'A', AC'B$ are together double of $B'C'A$ or BCA , therefore $A'BC'$ is the position of the triangle after revolving round the angles A, C .

Again, since the angles $C'BA', C'BA, CBA$, are all equal, CBC' is double of $A'BC'$ or ABC . Therefore the triangle, after revolving round its three angular points in succession, through angles double of the angles of the triangle at those points, returns to its initial position ABC .

ALGEBRA.

1848.

1. A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1 lb. a-head: after being at sea 20 days she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ lb. Find the original number of the crew.

Let x = the original number of the crew.

Then $60x$ = number of lbs. of biscuit with which they started.

$40x$ =remaining after being at sea
20 days.

And the remainder of the crew = $x - 5$, who have to remain at sea for 64 days, under a daily allowance of $\frac{5}{7}$ lb per man.

$$\therefore \frac{5}{7} 64 (x - 5) = 40x,$$

$$\therefore 64x - 320 = 56x,$$

$$\therefore 8x = 320,$$

$$\text{and } x = 40,$$

the original number of the crew.

2. If a , b , and x be positive, and $a > b$, prove that

$$\frac{x + a}{(x^2 + a^2)^{\frac{1}{2}}} > < \frac{x + b}{(x^2 + b^2)^{\frac{1}{2}}}, \text{ according as } x > < (ab)^{\frac{1}{2}}.$$

We have $\frac{x + a}{(x^2 + a^2)^{\frac{1}{2}}} > < \frac{x + b}{(x^2 + b^2)^{\frac{1}{2}}},$

$$\text{as } \frac{x^2 + 2ax + a^2}{x^2 + a^2} > < \frac{x^2 + 2bx + b^2}{x^2 + b^2},$$

$$\text{as } \frac{2ax}{x^2 + a^2} > < \frac{2bx}{x^2 + b^2},$$

as $\frac{a}{x^2 + a^2} > < \frac{b}{x^2 + b^2}$, since x is positive,
 as $(a - b) x^2 > < a^2 b - ab^2$,
 as $(a - b) x^2 > < ab(a - b)$,
 as $x^2 > < ab$, since a is greater than b ,
 or as $x > < (ab)^{\frac{1}{2}}$.

3. If a, b, c , be in harmonic progression, and n be a positive integer, shew that $a^n + c^n > 2b^n$.

Suppose a, b, c , all positive.* Then we have

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, in arithmetic progression,

$$\therefore \frac{2ac}{b} = a + c,$$

$$\text{but } a + c > 2\sqrt{ac},$$

$$\therefore \frac{2ac}{b} > 2\sqrt{ac},$$

$$\text{or } \sqrt{ac} > b,$$

$$\therefore a^{\frac{1}{2}n} a^{\frac{1}{2}n} > b^n,$$

$$\text{but } a^n + c^n > 2a^{\frac{1}{2}n} c^{\frac{1}{2}n},$$

$$a \text{ fortiori } a^n + c^n > 2b^n.$$

1849.

1. Reduce to its simplest form the expression

$$\frac{(1 - a^2)(1 - b^2)(1 - c^2) - (a + bc)(b + ca)(c + ab)}{1 - a^2 - b^2 - c^2 - 2abc}.$$

We have $(1 - a^2)(1 - b^2)(1 - c^2)$

$$= 1 - a^2 - b^2 - c^2 + b^2 c^2 + c^2 a^2 + a^2 b^2 - a^2 b^2 c^2$$

$$(a + bc)(b + ca)(c + ab)$$

$$= abc + b^2 c^2 + c^2 a^2 + a^2 b^2 + (a^2 + b^2 + c^2) abc + a^2 b^2 c^2;$$

* If one of these three quantities were negative, the proposition enunciated would not be necessarily true.

$$\begin{aligned}
\therefore (1-a^2)(1-b^2)(1-c^2) - (a+bc)(b+ca)(c+ab) \\
= 1-a^2-b^2-c^2-abc - (a^2+b^2+c^2)abc - 2a^2b^2c^2 \\
= (1-a^2-b^2-c^2-2abc)(1+abc), \\
\therefore \frac{(1-a^2)(1-b^2)(1-c^2) - (a+bc)(b+ca)(c+ab)}{1-a^2-b^2-c^2-2abc} = 1+abc.
\end{aligned}$$

2. Find a whole number which is greater than three times the integral part of its square root by unity. Shew that there are two solutions of the problem, and no more.

Let x be the integral part of the square root,
 $x^2 + y$ the whole number.

Then, by the conditions of the problem,

$$\begin{aligned}
x^2 + y &= 3x + 1, \\
\therefore x^2 - 3x &= 1 - y, \\
\therefore x^2 - 3x + \frac{9}{4} &= \frac{13}{4} - y, \\
\therefore x &= \frac{3}{2} \pm (\frac{13}{4} - y)^{\frac{1}{2}}.
\end{aligned}$$

Now y is essentially positive, and in order that x may be real it is necessary that y be less than $\frac{13}{4}$. Also, in order that x may be an integer, $(\frac{13}{4} - y)^{\frac{1}{2}}$ must be of the form $\frac{1}{2}(2m+1)$, m being some integer, such that $(2m+1)^2$ is less than 13. Therefore the only admissible values of m are 0 and 1. Hence $(\frac{13}{4} - y)^{\frac{1}{2}} = \frac{1}{2}$ or $\frac{3}{2}$, and

$$\begin{aligned}
x &= \frac{3}{2} + \frac{1}{2} \text{ or } \frac{3}{2} + \frac{3}{2}, \\
&= 2 \text{ or } 3.
\end{aligned}$$

Therefore $3x + 1 = 7$ or 10 ,

and 7 and 10 are the only solutions of the problem.

1850.

1. A number of persons were engaged to do a piece of work which would have occupied them m hours if they had commenced at the same time; but instead of doing so, they commenced at equal intervals, and then continued to work till the whole was finished, the payment being proportional to the work done by each; the first comer received r times as much as the last: find the time occupied.

Let x be the number of persons employed, y the number of hours the last worked, z the interval, in hours, between the first and second person commencing work, which is also that between the second and third, and so on. Then, by the conditions of the problem,

$$y + (x - 1) z = ry \dots\dots\dots (1).$$

Also, if W be the work done in an hour,

$[y + (y+z) + (y+2z) + \dots + \{y + (x-1) z\}] W =$ whole work done,

$$\therefore y + (y+z) + (y+2z) + \dots + \{y + (x-1) z\} = mx,$$

$$\text{or } \{2y + (x-1) z\} \frac{1}{2} x = mx,$$

$$\therefore 2y + (x-1) z = 2m,$$

$$\text{and, by (1), } y + (x-1) z = ry,$$

$$\therefore y = 2m - ry,$$

$$\text{or } y = \frac{2m}{r+1} \dots\dots (2).$$

Therefore whole time occupied = time from the first person beginning to work till the completion of the work

$$= y + (x-1) z,$$

$$= ry \text{ by (1),}$$

$$= \frac{2mr}{r+1} \text{ hours by (2).}$$

2. Shew that the product of the terms of an arithmetical progression is greater than $(al)^{\frac{1}{2}n}$; and that the sum of the terms of a geometrical progression is less than $(a+l) \frac{1}{2}n$; where in both cases a , l , and n denote the first and last terms, and the number of terms respectively.

(α) Let b be the common difference in the arithmetical progression, P the product of its terms, then

$$P = a(a+b)(a+2b) \dots \{a + (n-1)b\},$$

$$\text{and } l = a + (n-1)b.$$

$$\text{Now } (a+mb) \{a + (n-m-1)b\} = a^2 + (n-1)ab + (n-m-1)mb^2.$$

This will be least when $m = 0$, negative values of m being excluded; but in that case

$$(a + mb) \{a + (n - m - 1) b\} = al;$$

therefore the product of any two terms equidistant from the mean is not less than al .

The property enunciated follows at once from this when n is even. If n is odd, the middle term is

$$a + \frac{n-1}{2} b,$$

$$\begin{aligned} \text{and } \left(a + \frac{n-1}{2} b\right)^2 &= a^2 + (n-1) ab + \left(\frac{n-1}{2} b\right)^2, \\ &> a^2 + (n-1) ab, \\ &> al; \end{aligned}$$

$$\therefore a + \frac{n-1}{2} b > (al)^{\frac{1}{2}}.$$

Hence, in all cases, $P > (al)^{\frac{1}{2}n}$.

(β) Let r be the common ratio in the geometrical progression, S the sum of its terms; then

$$S = a \frac{r^n - 1}{r - 1},$$

$$\text{and } l = ar^{n-1},$$

$$\therefore a + l = a(1 + r^{n-1}).$$

$$\begin{aligned} \text{Now } ar^m + ar^{n-m-1} - (a + l) &= a(r^m - 1) - a(r^{n-1} - r^{n-m-1}), \\ &= a(r^m - 1)(1 - r^{n-m-1}), \end{aligned}$$

which is negative for all positive values of m .

Hence the sum of the first and last terms is greater than the sum of any other two terms equidistant from the mean.

The property enunciated follows at once from this when n is even. If n be odd, the middle term is

$$ar^{\frac{1}{2}(n-1)},$$

$$\text{and } r^{\frac{1}{2}(n-1)} < \frac{1 + r^{n-1}}{2},$$

$$\therefore ar^{\frac{1}{2}(n-1)} < \frac{a + l}{2}.$$

Hence, in all cases, $S < (a + l)^{\frac{1}{2}n}$.

1851.

1. If $\frac{3x-2}{(x-1)(x-2)(x-3)}$ be expanded in a series ascending by powers of x , find the coefficient of x^n .

This is best effected by resolving the given expression into its partial fractions, and expanding each fraction separately.

For this purpose, assume

$$\frac{3x-2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3},$$

A, B, C being independent of x .

$$\text{Then } 3x-2 = A(x-2)(x-3) + B(x-3)(x-1) + C(x-1)(x-2)$$

Hence, putting $x = 1$, *identically.*

$$3 \cdot 1 - 2 = A(1-2)(1-3),$$

$$\text{or } 1 = 2A \dots \dots \dots (1).$$

Similarly, putting $x = 2$,

$$3 \cdot 2 - 2 = B(2-3)(2-1),$$

$$\text{or } 4 = -B \dots \dots \dots (2),$$

and putting $x = 3$,

$$3 \cdot 3 - 2 = C(3-1)(3-2),$$

$$\text{or } 7 = 2C \dots \dots \dots (3);$$

$$\begin{aligned} \therefore \frac{3x-2}{(x-1)(x-2)(x-3)} &= \frac{1}{2(x-1)} - \frac{4}{x-2} + \frac{7}{2(x-3)}, \\ &= -\frac{1}{2} \frac{1}{1-x} + \frac{2}{1-\frac{1}{2}x} - \frac{7}{6} \frac{1}{1-\frac{1}{3}x}, \\ &= -\frac{1}{2}(1+x+\dots+x^n+\dots) \\ &\quad + 2\{1+\frac{1}{2}x+\dots+(\frac{1}{2}x)^n+\dots\} \\ &\quad - \frac{7}{6}\{1+\frac{1}{3}x+\dots+(\frac{1}{3}x)^n+\dots\}, \end{aligned}$$

in which the coefficient of x^n

$$= -\frac{1}{2} + \frac{2}{2^n} - \frac{7}{6} \frac{1}{3^n},$$

$$= \frac{1}{2^{n-1}} - \frac{1}{2} - \frac{7}{2} \frac{1}{3^{n+1}},$$

the required coefficient.

2. Find the sum of the different numbers which can be formed with m digits α , n digits β , &c., the entire series of $m + n + \&c.$ digits being employed in the formation of each number.

The total number of numbers which can thus be formed, is equal to the number of permutations of $m + n + \dots$ things, whereof m are of one kind, n of another..., taken all together,

$$= \frac{1.2\dots(m+n+\dots)}{(1.2\dots m)(1.2\dots n)\dots}$$

Now, the number of times α will be found in any assigned place: the number of times β will be found there: $\dots :: m : n : \dots$; therefore the number of times α will be found there

$$= \frac{m}{m+n+\dots} \frac{1.2\dots(m+n+\dots)}{(1.2\dots m)(1.2\dots n)\dots} = m \frac{1.2\dots(m+n+\dots-1)}{(1.2\dots m)(1.2\dots n)\dots}$$

Similar expressions holding for the number of times β , γ ,... will be found there: we have, if S be the sum of the digits in any assigned place,

$$S = \frac{(m\alpha + n\beta + \dots) 1.2\dots(m+n+\dots-1)}{(1.2\dots m)(1.2\dots n)\dots};$$

therefore if Σ denote the sum of all the numbers,

$$\begin{aligned} \Sigma &= S(1 + 10 + 10^2 + \dots + 10^{m+n+\dots-1}), \\ &= S \frac{10^{m+n+\dots} - 1}{9}, \\ &= \frac{10^{m+n+\dots} - 1}{9} \frac{(m\alpha + n\beta + \dots) 1.2\dots(m+n+\dots-1)}{(1.2\dots m)(1.2\dots n)\dots}, \end{aligned}$$

the sum required.

3. The difference between the arithmetic and geometric means of two numbers is less than one-eighth of the squared difference of the numbers divided by the less number, but greater than one-eighth of such squared difference divided by the greater number. If x, y be any two numbers, x_1, y_1 their arithmetic and geometric means, x_2, y_2 the arithmetic and geometric means of x_1, y_1 , and so on, find major and minor limits for the difference $x_n - y_n$.

(α) Taking the notation of the latter part of the problem, we have

$$x_1 = \frac{x+y}{2}, \quad y_1 = (xy)^{\frac{1}{2}},$$

$$\therefore x_1 - y_1 = \frac{x - 2(xy)^{\frac{1}{2}} + y}{2} = \frac{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2}{2}.$$

Now, one-eighth of the squared difference of the numbers

$$= \frac{1}{8}(x-y)^2$$

$$= \frac{(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2 (x^{\frac{1}{2}} + y^{\frac{1}{2}})^2}{8},$$

$$= (x_1 - y_1) \frac{(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2}{4}.$$

And $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2$ lies between $(2y^{\frac{1}{2}})^2$ and $(2x^{\frac{1}{2}})^2$,

$$\text{or } > 4y, \quad < 4x;$$

$$\therefore x_1 - y_1 = \frac{4}{(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2} \frac{1}{8}(x-y)^2,$$

$$< \frac{1}{y} \frac{(x-y)^2}{8},$$

$$> \frac{1}{x} \frac{(x-y)^2}{8}.$$

(β) From above it appears that

$$x_n - y_n < \frac{(x_{n-1} - y_{n-1})^2}{8y_{n-1}}, \quad > \frac{(x_{n-1} - y_{n-1})^2}{8x_{n-1}};$$

therefore, *a fortiori*,

$$x_n - y_n < \frac{1}{8y_{n-1}} \frac{(x_{n-2} - y_{n-2})^4}{(8y_{n-2})^2}, \quad > \frac{1}{8x_{n-1}} \frac{(x_{n-2} - y_{n-2})^2}{(8x_{n-2})^2},$$

$$< \dots\dots\dots, \quad > \dots\dots\dots,$$

$$< \frac{(x-y)^{2^n}}{8y_{n-1}(8y_{n-2})^2 \dots (8y)^{2^{n-1}}}, \quad > \frac{(x-y)^{2^n}}{8x_{n-1}(8x_{n-2})^2 \dots (8x)^{2^{n-1}}}.$$

4. If all the sums of two letters that can be formed with any n letters be multiplied together, then in each term of the product, the sum of any r of the indices cannot exceed the number $rn - \frac{1}{2}r(r+1)$.

Let a_1, a_2, \dots, a_n , be the letters.

Then the product will be of the form

$$(a_1 + a_2) (a_1 + a_3) \dots (a_1 + a_n) (a_2 + a_3) \dots (a_{n-1} + a_n).$$

Now, any one of the letters, as a_1 , only appears $n - 1$ times in this product, that is, once in each of the factors $a_1 + a_2 \dots a_1 + a_n$, therefore in no term can its index be greater than $n - 1$.

And in such term, the index of a_2 cannot be greater than $n - 2$, for a_2 can only enter $n - 2$ times as a factor of such a term, the factor $a_1 + a_2$ being excluded.

Similarly, in the term involving

$$a_1^{n-1} a_2^{n-2} \text{ the index of } a_3 \text{ cannot be } > n - 3,$$

$$\dots \dots \dots a_1^{n-1} a_2^{n-2} \dots a_{r-1}^{n-r+1} \dots a_r \dots > n - r,$$

therefore the sum of any r of the indices cannot be greater than

$$\begin{aligned} & (n - 1) + (n - 2) + \dots + (n - r), \\ & = \{2n - 2 - (r - 1)\} \frac{1}{2}r, \\ & = rn - \frac{1}{2}r(r + 1), \end{aligned}$$

the required limit, which the sum of the r indices cannot exceed.

5. Eliminate x from the equations

$$(x - a) (x - b) = (x - c) (x - d) = (x - e) (x - f),$$

and from the same equations with the additional relation $e = f$, find a quadratic equation for determining the quantity e or f . Shew also that if m', m'' be the values of e or f , then $m'' - m'$ is a harmonic mean between $a - m'$, $b - m'$, and between $c - m'$, $d - m'$.

(a) Since $(x - a) (x - b) = (x - c) (x - d) = (x - e) (x - f)$, we get

$$x^2 - (a + b) x + ab = x^2 - (c + d) x + cd = x^2 - (e + f) x + ef;$$

$$\therefore (e + f - a - b) x = ef - ab,$$

$$(e + f - c - d) x = ef - cd,$$

$$\therefore (ef - cd) (e + f - a - b) = (ef - ab) (e + f - c - d) \dots (1),$$

C

an equation from which x is eliminated, and which may be put in the more symmetrical form

$$ab(c+d-e-f) + cd(e+f-a-b) + ef(a+b-c-d) = 0 \dots (2).$$

(β) If $e = f$, equation (1) becomes

$$(e^2 - cd)(2e - a - b) = (e^2 - ab)(2e - c - d),$$

$$\therefore (c+d-a-b)e^2 + 2(ab-cd)e + (a+b)cd - (c+d)ab = 0 \dots (3),$$

the quadratic for the determination of e or f .

(γ) If m' , m'' be the roots of equation (3),

$$m' + m'' = 2 \frac{ab - cd}{a + b - c - d},$$

$$m'm'' = \frac{ab(c+d) - cd(a+b)}{a + b - c - d};$$

$$\begin{aligned} \therefore 2(ab + m'm'') &= 2 \frac{(ab - cd)(a+b)}{a + b - c - d}, \\ &= (a+b)(m' + m''). \end{aligned}$$

Now

$$2(ab + m'm'') - (a+b)(m' + m'') = (b-m')(a-m'') - (a-m')(m''-b),$$

$$\text{whence } (a-m')(m''-b) = (b-m')(a-m''),$$

$$\therefore a - m' : b - m' :: a - m'' : m'' - b,$$

$$\text{or } a - m' : b - m' :: a - m' - (m'' - m') : m'' - m' - (b - m'),$$

whence $m'' - m'$ is a harmonic mean between $a - m'$, $b - m'$.

Similarly, it is a harmonic mean between $c - m'$, $d - m'$.

TRIGONOMETRY.

1848.

1. The angles of a quadrilateral inscribed in a circle taken in order, when multiplied by 1, 2, 2, 3, respectively, are in Arithmetical Progression; find their values.

Let θ, ϕ be two adjacent angles of the quadrilateral, then $\pi - \theta, \pi - \phi$, will be the angles respectively opposite to them; and, by the conditions of the problem, $\theta, 2\phi, 2(\pi - \theta), 3(\pi - \phi)$, are in Arithmetical Progression.

$$\therefore 2\phi - \theta = 2(\pi - \theta) - 2\phi = 3(\pi - \phi) - 2(\pi - \theta),$$

$$\therefore 4\phi + \theta = 2\pi,$$

$$4\theta - \phi = \pi,$$

$$\therefore 17\theta = 6\pi,$$

$$17\phi = 7\pi,$$

$$\therefore \theta = \frac{6\pi}{17}, \quad \phi = \frac{7\pi}{17},$$

$$\pi - \theta = \frac{11\pi}{17}, \quad \pi - \phi = \frac{10\pi}{17},$$

the required values of the angles of the quadrilateral.

2. Prove that $\sin 3\theta \sin^3 \theta + \cos 3\theta \cos^3 \theta = \cos^3 2\theta$.

We have $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$,

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta;$$

$$\therefore \sin 3\theta \sin^3 \theta + \cos 3\theta \cos^3 \theta$$

$$= 3(\sin^4 \theta - \cos^4 \theta) + 4(\cos^6 \theta - \sin^6 \theta),$$

$$= 3(\cos^2 \theta + \sin^2 \theta)(\sin^4 \theta - \cos^4 \theta) + 4(\cos^6 \theta - \sin^6 \theta),$$

$$= \cos^6 \theta - 3 \cos^4 \theta \sin^2 \theta + 3 \cos^2 \theta \sin^4 \theta - \sin^6 \theta,$$

$$= (\cos^2 \theta - \sin^2 \theta)^3,$$

$$= \cos^3 2\theta,$$

the required result.

3. Having given the three right lines drawn from any point to the three angular points of an equilateral triangle, determine a side of the triangle.

Let ABC (fig. 16) be the triangle, O the point from which the lines are drawn, $OA = a$, $OB = b$, $OC = c$; also let the angle $BAO = \theta$, $CAO = \phi$, and let a side of the triangle $= x$. Then, by the triangle BAO ,

$$x^2 + a^2 - 2ax \cos \theta = b^2 \dots\dots\dots(1).$$

By the triangle CAO ,

$$x^2 + a^2 - 2ax \cos \phi = c^2 \dots\dots\dots(2).$$

Also

$$\theta + \phi = \frac{1}{3}\pi.$$

Adding (1) and (2), we get, observing that $\cos \theta + \cos \phi = 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2}$,

$$2(x^2 + a^2) - 4ax \cos \frac{1}{3}\pi \cos \frac{\theta - \phi}{2} = b^2 + c^2 \dots\dots(3).$$

Subtracting (2) from (1), and observing that $\cos \phi - \cos \theta = 2 \sin \frac{\theta + \phi}{2} \sin \frac{\theta - \phi}{2}$,

$$4ax \sin \frac{1}{3}\pi \sin \frac{\theta - \phi}{2} = b^2 - c^2 \dots\dots\dots(4).$$

By (3) and (4),

$$16a^2x^2 = \frac{\{b^2 + c^2 - 2(x^2 + a^2)\}^2}{\cos^2 \frac{1}{3}\pi} + \frac{(b^2 - c^2)^2}{\sin^2 \frac{1}{3}\pi},$$

$$\therefore \frac{\{b^2 + c^2 - 2(x^2 + a^2)\}^2}{3} + (b^2 - c^2)^2 = 4a^2x^2,$$

$$\therefore (b^2 + c^2 - 2a^2)^2 - 4x^2(b^2 + c^2 - 2a^2 + 3a^2) + 3(b^2 - c^2)^2 + 4x^4 = 0,$$

$$\therefore x^4 - (a^2 + b^2 + c^2)x^2 + a^4 + b^4 + c^4 - (b^2c^2 + c^2a^2 + a^2b^2) = 0,$$

$$\begin{aligned} \therefore x^2 &= \frac{a^2 + b^2 + c^2}{2} \pm \left\{ \frac{(a^2 + b^2 + c^2)^2}{4} - (a^4 + b^4 + c^4) + b^2c^2 + c^2a^2 + a^2b^2 \right\}^{\frac{1}{2}} \\ &= \frac{a^2 + b^2 + c^2}{2} \pm \frac{3^{\frac{1}{2}}}{2} \{2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4)\}^{\frac{1}{2}}, \end{aligned}$$

which determines a side of the triangle.

1849.

1. If $\phi - \alpha$, ϕ , $\phi + \alpha$, be three angles whose cosines are in Harmonical Progression, prove that

$$\cos \phi = 2^{\frac{1}{2}} \cos \frac{1}{2} \alpha.$$

Since $\cos(\phi - \alpha)$, $\cos \phi$, $\cos(\phi + \alpha)$ are in Harmonical Progression, we have

$$\begin{aligned} \frac{2}{\cos \phi} &= \frac{1}{\cos(\phi + \alpha)} + \frac{1}{\cos(\phi - \alpha)}, \\ &= \frac{2 \cos \phi \cos \alpha}{\cos(\phi - \alpha) \cos(\phi + \alpha)}; \end{aligned}$$

$$\begin{aligned} \therefore \cos^2 \phi \cos \alpha &= \cos(\phi - \alpha) \cos(\phi + \alpha), \\ &= \frac{1}{2}(\cos 2\phi + \cos 2\alpha), \\ &= \cos^2 \phi - \sin^2 \alpha, \end{aligned}$$

$$\begin{aligned} \therefore \cos^2 \phi &= \frac{\sin^2 \alpha}{1 - \cos \alpha}, \\ &= \frac{4 \sin^2 \frac{1}{2} \alpha \cos^2 \frac{1}{2} \alpha}{2 \sin^2 \frac{1}{2} \alpha}, \\ &= 2 \cos^2 \frac{1}{2} \alpha, \end{aligned}$$

$$\therefore \cos \phi = 2^{\frac{1}{2}} \cos \frac{1}{2} \alpha,$$

the required relation.

2. A person wishing to ascertain his distance from an inaccessible object, finds three points in the horizontal plane at which the angular elevation of the summit of the object is the same. Shew how the distance may be found.

Let O (fig. 17) be the foot of the object; A , B , C the three points at which the angular elevation of the summit of the object is the same; then they must all be at the same distance from O . Let x be this common distance.

Let the angle $AOB = \theta$, the angle $AOC = \phi$. Measure BC , CA , AB , and let their distances = a , b , c , respectively;

then

$$a = 2x \sin \frac{1}{2}(\theta + \phi),$$

$$b = 2x \sin \frac{1}{2}\phi,$$

$$c = 2x \sin \frac{1}{2}\theta.$$

Eliminate θ , ϕ , from these three equations, then x will be known and the distance of the person from the object determined.

1850.

A person wishing to ascertain the distances between three inaccessible objects A , B , C , (fig. 18), places himself in a line with A and B ; he then measures the distances along which he must walk in a direction at right angles to AB , until A , C , and B , C , respectively, are in a line with him, and also observes in those positions their angular bearings: shew how he can find the distances between A , B , and C .

Let DE and DF , the measured distances, $= d$ and e ; BEA and BFA the observed angles $= \alpha$ and β . Let the sides of the triangle $ABC = a$, b , c , and $BD = x$.

$$\text{Therefore } \tan DEA = \frac{x + c}{d},$$

$$\text{and } \tan DEB = \frac{x}{d},$$

$$\therefore \tan \alpha = \tan(DEA - DEB),$$

$$\begin{aligned} &= \frac{\frac{x+c}{d} - \frac{x}{d}}{1 + \frac{x(x+c)}{d^2}}, \\ &= \frac{cd}{d^2 + x(x+c)} \dots\dots\dots (1). \end{aligned}$$

$$\text{Similarly, } \tan \beta = \frac{ce}{e^2 + x(x+c)} \dots\dots\dots (2).$$

From equations (1) and (2), x and c are known, and thence

$$BAC = \tan^{-1} \frac{d}{x+c}, \quad \text{and} \quad CBA = 180^\circ - \tan^{-1} \frac{e}{x},$$

and thence the distances a and b .

1851.

1. If $\tan \beta = \frac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha},$

shew that $\tan(\alpha - \beta) = (1 - n) \tan \alpha.$

$$\begin{aligned} \text{We have } \tan(\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}, \\ &= \frac{\tan \alpha - \frac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}}{1 + \frac{n \sin^2 \alpha}{1 - n \sin^2 \alpha}}, \\ &= \tan \alpha - n \frac{\sin^3 \alpha}{\cos \alpha} - n \sin \alpha \cos \alpha, \\ &= \frac{\sin \alpha - n(\sin^2 \alpha + \cos^2 \alpha) \sin \alpha}{\cos \alpha}, \\ &= (1 - n) \tan \alpha. \end{aligned}$$

2. Two triangles stand on the same base, determine in terms of the base and of the tangents of the angles at the base, the distance between the vertices of the triangles.

Let $ABC, A'BC$ (fig. 19) be the two triangles. Let BC the base $= a$, and let the angles $ABC, ACB = B, C$, respectively, and the angles $A'BC, A'CB = B', C'$. Join AA' , and let $AA' = r$, then it is required to find the magnitude of r .

Draw $AD, A'D'$ perpendicular to the base. Then

$$\begin{aligned} r^2 &= (AD - A'D')^2 + DD'^2, \\ &= (AD - A'D')^2 + (A'D' \cot B' - AD \cot B)^2. \end{aligned}$$

Now $a = AD(\cot B + \cot C),$
also $= A'D'(\cot B' + \cot C');$

$$\begin{aligned} \therefore r^2 &= a^2 \left(\frac{1}{\cot B + \cot C} - \frac{1}{\cot B' + \cot C'} \right)^2 \\ &\quad + a^2 \left(\frac{\cot B'}{\cot B' + \cot C'} - \frac{\cot B}{\cot B + \cot C} \right)^2, \end{aligned}$$

$$\text{or } r = a \left\{ \left(\frac{\tan B \tan C}{\tan B + \tan C} - \frac{\tan B' \tan C'}{\tan B' + \tan C'} \right)^2 + \left(\frac{\tan C'}{\tan B' + \tan C'} - \frac{\tan C}{\tan B + \tan C} \right)^2 \right\}^{\frac{1}{2}},$$

an expression of the required form.

CONIC SECTIONS.

1848.

1. Given the lengths of the axes of an ellipse, and the positions of one focus, and of one point in the curve: give a geometrical construction for finding the centre.

Let MN (fig. 20) be a line equal in length to the axis-minor. With N as centre and a radius equal to the axis-major, describe an arc of a circle. From M draw MO perpendicular to MN , and cutting the arc in O , MO will be equal to the distance between the foci of the ellipse.

Produce SP to Q (fig. 21) making SQ equal to NO . With P as centre, and PQ as radius, describe an arc of a circle, and with S as centre, and radius equal to MO , describe another arc; H the point of intersection of these arcs will be the other focus, for SP , PH are together equal to the axis-major, and SH is equal to the distance between the foci. If therefore we bisect SH in C , C will be the centre.

Since the arcs described from S , P as centres will in general intersect in two points, it appears that there are two positions which the centre may have.

2. P is any point in an ellipse (fig. 22), AA' its axis-major, NP an ordinate to the point P ; to any point Q in the curve draw AQ , $A'Q$, meeting NP in R and S ; shew that

$$NR.NS = NP^2.$$

Draw the ordinate QM , then by similar triangles ANR , AMQ ,

$$NR : NA :: MQ : MA,$$

and by similar triangles $A'NS$, $A'MQ$,

$$NS : NA' :: MQ : MA',$$

$$\text{therefore} \quad NR.NS : NA.NA' :: MQ^2 : MA.MA', \\ \qquad \qquad \qquad :: BC^2 : AC^2;$$

$$\text{also} \quad NP^2 : NA.NA' :: BC^2 : AC^2,$$

$$\text{therefore} \quad NR.NS = NP^2.$$

3. PSp (fig. 23) is a focal chord of a parabola, RDr the directrix meeting the axis in D ; Q is any point in the curve: prove that if PQ, pQ produced meet the directrix in R, r , half the latus-rectum will be a mean proportional between DR, Dr .

Draw PN, Qm perpendicular to the directrix, and join SR, Sr, SQ .

$$\begin{aligned} \text{Then} \quad \sin PRS &= \sin PSR \frac{PS}{PR} \\ &= \sin PSR \frac{PN}{PR} \\ &= \sin PSR \cdot \sin PRN : \end{aligned}$$

$$\text{similarly} \quad \sin QRS = \sin QSR \cdot \sin QRN,$$

$$\therefore \sin PSR = \sin QSR,$$

$$\therefore QSR = pSR :$$

$$\text{similarly} \quad QSr = PSr,$$

$$\therefore RSr \text{ is a right angle,}$$

$$\therefore SD^2 = DR.Dr,$$

or half the latus-rectum is a mean proportional between DR, Dr .

1849.

1. Draw a parabola to touch a given circle in a given point, so that its axis may touch the same circle in another given point.

Let PQR (fig. 24) be the given circle, P the point in which the parabola is to touch it, Q the point in which the axis is to touch it. Draw PT a tangent to the circle at P , this will also be a tangent to the parabola at P . Draw QT touching the circle at Q , and at the point P in the straight line TP , make the

angle TPS equal to the angle PTQ ; then S , the intersection of QT and PS , will be the focus of the parabola. Bisect TP in K , and draw KA perpendicular to ST . A will be the vertex of the parabola, and the vertex and focus being found, the curve may be constructed.

2. If a circle be described touching the axis-major of an ellipse in one of the foci, and passing through one extremity of the axis-minor, the semiaxis-major will be a mean proportional between the diameter of this circle and the semiaxis-minor.

Let AA' (fig. 25) be the axis-major of the ellipse, S the focus, C the centre, B the extremity of the axis-minor. Describe the circle SBP touching AA' in S , and passing through B , and draw the diameter SP .

Join SB , BP ; then the angle SBP , being in a semicircle, is a right angle, also the angle SCB is a right angle. And the angle BSP is equal to the angle SBC , therefore the triangles PSB , SBC are similar. Hence

$$BC : SB :: SB : SP,$$

or SB is a mean proportional between SP and BC . But SB is equal to the semiaxis-major; therefore the semiaxis-major is a mean proportional between the diameter of the circle and the semiaxis-minor.

3. If AB , CD , two lines in an ellipse, not parallel to one another, make equal angles with either axis; the lines AC , BD and AD , BC will also make equal angles with either axis.

Let $A'B'C'D'$ (fig. 26) be the points of intersection of the perpendiculars to the axis-major through the points $ABCD$ with the auxiliary circle $A'B'C'D'$. Then it is evident that a line joining any two of the above points as $A'B'$ will intersect the axis-major in the same point as AB does, and any two lines joining the above points as $A'B'$, $C'D'$ will be equally inclined to the axis-major, and therefore to either axis, if AB , CD are so,

and *vice versâ*: hence we have to prove that if $A'B'$, $C'D'$ are equally inclined to the axis-major LFE , the lines $A'C'$, $B'D'$ and $A'D'$, $B'C'$ are so.

Now $\angle A'GL = \angle A'EL + \angle B'A'C'$,
 and $\angle D'HL = \angle D'FL + \angle B'D'C'$;
 also $\angle A'EL = \angle D'FL$,
 and $\angle B'A'C' = \angle B'D'C'$;
 therefore $\angle A'GL = \angle D'HL = \angle B'HG$,
 or $A'C'$, $B'D'$ are equally inclined to LFE .

Again, $\angle A'LH = \angle LD'H + \angle LHD'$,
 and $\angle B'KG = \angle KC'G + \angle KGC'$;
 also $\angle LD'H = \angle KC'G$,
 and $\angle LHD' = \angle KGC'$;
 therefore $\angle A'LH = \angle B'KG$,

or $A'D'$ and $B'C'$ are also equally inclined to LFE ; therefore also AC , BD and AD , BC are equally inclined to either axis.
 Q. E. D.

1850.

1. If from any point P of a circle, PC be drawn to the centre C , and a chord PQ be drawn parallel to the diameter ACB , and bisected in R , shew that the locus of the intersection of CP and AR is a parabola.

Let O (fig. 27) be the intersection of CP and AR . Draw AM , CN perpendicular to AB , ONM parallel to AB . CN will pass through R . Then

$$CO : CP :: AO : AR, \\ :: OM : MN.$$

But $CP = AC = MN$,
 therefore $CO = OM$,

and the locus of O is a parabola, of which C is the focus, AM the directrix.

2. From the point P in the ellipse APB (fig. 28), lines are drawn to A, B , the extremities of the axis-major, and from A, B , lines are drawn perpendicular to AP, BP ; shew that the locus of their intersection will be another ellipse, and find its axes.

Let Q be the intersection of the lines perpendicular to AP, BP . Draw PM, QN perpendicular to the axis-major, then the triangles PBM, BQN will be similar, therefore

$$PM : BM :: BN : QN;$$

similarly $PM : AM :: AN : QN,$

therefore $PM^2 : AM.BM :: AN.BN : QN^2.$

But if bC be the semiaxis-minor of the original ellipse,

$$PM^2 : AM.BM :: bC^2 : AC^2,$$

therefore $AN.BN : QN^2 :: bC^2 : AC^2;$

therefore the locus of Q is an ellipse whose axes are to one another as $bC : AC$.

And if we draw AB', BB' perpendicular to Ab, Bb , we have

$$B'C = \frac{AC^2}{bC},$$

which is one axis, the other is equal to $\frac{bC}{AC} B'C$ or AC .

3. If two ellipses having the same major axes, can be inscribed in a parallelogram, the foci of the ellipses will lie in the corners of an equiangular parallelogram.

For it is evident that the centres of the ellipses must lie at the point of intersection of the diagonals of the parallelogram, that is, must be coincident, and their major axes are equal; therefore they will have a common auxiliary circle.

The lines joining the points of intersection of this circle and the parallelogram, will, if the right points are joined, be perpendicular to the sides of the parallelogram, and each of them will contain two foci: hence the four foci will be at their points of intersection, that is, at the corners of a parallelogram, equiangular with the circumscribing parallelogram.

4. If from the extremities of any diameter AB (fig. 29) of an equilateral hyperbola, lines be drawn to any point P in the curve, they will be equally inclined to the asymptotes.

From A and P draw the perpendiculars AC , AF , PE , PF , on the asymptotes, AF and PF intersecting in F ; from B draw BD perpendicular to the asymptote Oq . Then, since $PQ = Bq$, and the triangles QEP , BDq are similar, they are also equal; therefore $PE = Dq$ and $QE = BD$, therefore

$$AF = CO + EP = OD + Dq = Oq,$$

and $PF = OE + AC = OE + BD = OE + EQ = OQ$;

therefore the right-angled triangles APF , qQO are equal in all respects, and the chords PA , PB equally inclined to the asymptotes.

1851.

1. Given a pair of conjugate diameters of a conic section, find geometrically the position of the principle diameters, (1) in the case of the hyperbola, (2) in that of the ellipse.

Let PP' , DD' (fig. 30) be the given conjugate diameters of an hyperbola intersecting in C .

Join PD , PD' ; bisect them in E and F , draw CE , CF ; bisect the angle ECF by the line $A'CA$, and through C draw BCB' perpendicular to ACA' ; these will be the principal diameters required. For, by the property of the hyperbola, CE , CF are the asymptotes, and ACA' , BCB' , to which they are equally inclined, are the principal diameters.

2.* The solution of this part of the problem depends upon the property of the ellipse, that if P (fig. 31) be any point in the ellipse, and CR , CN , two lines at right angles to each other, cut the straight line PRN in the points R , N , such that PN is equal to the semiaxis-major, and PR to the semiaxis-minor, CR and CN will be the directions of the principal axes.

* For this solution the authors are indebted to the kindness of the Moderator, Mr. Gaskin.

Let CP, CD be the given semi-conjugate diameters; draw PF perpendicular to CD ; make PK equal to CD ; upon CK as diameter describe the circle CFK ; through its centre O draw PRN ; join CR, CN : these will be the directions of the principal axes.

For $PF.CD = PF.PK = PR.PN = AC.BC$,

$$\text{and } CP^2 + CD^2 = CP^2 + PK^2 = 2CO^2 + 2PO^2 = 2OR^2 + 2OP^2, \\ = PR^2 + PN^2 \text{ (} Euc. II. 10 \text{)} = AC^2 + BC^2,$$

therefore $PN = AC, PR = BC$,

and CR, CN are the directions of the principal axes.

STATICS.

1849.

1. Two forces F and F' , acting in the diagonals of a parallelogram, keep it at rest in such a position that one of its edges is horizontal; shew that $F \sec \alpha = F' \sec \alpha' = W \operatorname{cosec}(\alpha + \alpha')$, where W is the weight of the parallelogram, α and α' the angles between its diagonals and the horizontal side.

Let AB (fig. 32) be the horizontal side of the parallelogram. In order to preserve equilibrium, the directions of the forces F , F' must meet in G , the centre of gravity. Hence, by the triangle of force,

$$\frac{F}{\sin BGW} = \frac{F'}{\sin AGW} = \frac{W}{\sin AGB},$$

or
$$\frac{F}{\cos \alpha} = \frac{F'}{\cos \alpha'} = \frac{W}{\sin(\alpha + \alpha')};$$

therefore
$$F \sec \alpha = F' \sec \alpha' = W \operatorname{cosec}(\alpha + \alpha').$$

2. A cubical box is half-filled with water, and placed upon a rough rectangular board; if the board be slowly inclined to the horizon, determine whether the box will slide down or topple over.

Let μ = the coefficient of friction.

Then the box would begin to slide when the inclination of the board to the horizon $= \tan^{-1} \mu$.

It would begin to topple when the inclination $= \frac{1}{4} \pi$.

Therefore it will begin to slide or topple over, according as $\mu < \text{or} > 1$.

1850.

1. A heavy body is supported in a given position by means of a string which is fastened to two given points in the body,

and then passes over a smooth peg: find the length of the string.

Let G (fig. 33) be the given position of the centre of gravity, A and B those of the points of support. The position of the peg P is determined by the conditions that it must lie in the vertical through G , and that the angles APG , BPG must be equal, each $= \theta$ suppose.

Let $AG = a$, $BG = b$, $\angle AGP = \alpha$, $\angle BGP = \beta$; then

$$\frac{PG}{AG} = \frac{\sin PAG}{\sin APG} = \frac{\sin(\theta + \alpha)}{\sin \theta} = \cos \alpha + \sin \alpha \cot \theta;$$

similarly
$$\frac{PG}{BG} = \cos \beta + \sin \beta \cot \theta,$$

therefore
$$\frac{\cos \alpha + \sin \alpha \cot \theta}{\cos \beta + \sin \beta \cot \theta} = \frac{BG}{AG} = \frac{b}{a},$$

whence θ is known, and length of the string

$$\begin{aligned} &= AP + BP, \\ &= \frac{\sin \alpha}{\sin \theta} a + \frac{\sin \beta}{\sin \theta} b \text{ is known.} \end{aligned}$$

2. Two spheres are supported by strings attached to a given point, and rest against one another: find the tensions of the strings.

Let A , B , (fig. 34) be the centres of the spheres, and C the peg. Then, if the spheres are smooth, the strings must lie in the lines CA , CB ; hence the parts of the triangle ABC are known. To determine its position.

Let G be the centre of gravity of the spheres, CG must be vertical. Let W_1 , W_2 be the weights of the spheres A and B , therefore

$$AG : BG :: W_2 : W_1;$$

and if $\angle ACG = \theta$,

$$\frac{\sin(C - \theta)}{\sin \theta} = \frac{BG \sin B}{AG \sin A} = \frac{W_1 \sin B}{W_2 \sin A};$$

therefore
$$\sin C \cot \theta - \cos C = \frac{W_1 \sin B}{W_2 \sin A},$$

whence θ is known.

Let T_1 , T_2 be the tensions of the strings which support A and B respectively; therefore resolving the forces on the sphere A perpendicular to AB ,

$$T_1 \sin A - W_1 \sin(A + \theta) = 0,$$

or
$$T_1 = \frac{\sin(A + \theta)}{\sin A} W_1;$$

and similarly
$$T_2 = \frac{\sin(B + \overline{C - \theta})}{\sin B} W_2,$$

$$= \frac{\sin(A + \theta)}{\sin B} W_2,$$

whence T_1 and T_2 are known.

3. A cone of given weight W (fig. 35) is placed with its base on a smooth inclined plane, and supported by a weight W' , which hangs by a string fastened to the vertex of the cone, and passing over a pulley in the inclined plane at the same height as the vertex. Find the angle of the cone when the ratio of the weights is such that a small increase of W' would cause the cone to turn about the highest point of the base, as well as slide.

Let α = the angle of the plane,

θ = the half-angle of the cone.

Since the resolved parts, along the plane, of the tension W' of the string and the weight W just balance, we have

$$W \sin \alpha = W' \cos \alpha \dots \dots \dots (1);$$

and because the moments of the same forces about B are also equal,

$$W \sin \alpha \cdot \frac{1}{4} AC + W \cos \alpha \cdot BC = W' \cos \alpha \cdot AC - W' \sin \alpha \cdot BC,$$

$$W' \cos \alpha \cdot \frac{3}{4} AC = \left(W' \frac{\cos^2 \alpha}{\sin \alpha} + W' \sin \alpha \right) BC, \text{ from (1),}$$

or
$$BC = \frac{3}{4} \sin \alpha \cos \alpha AC,$$

or
$$\tan \theta = \frac{3}{8} \sin 2\alpha.$$

1851.

1. A cone whose semi-vertical angle is $\tan^{-1} \frac{1}{2}$ is enclosed in the circumscribing spherical surface, shew that it will rest in any position.

Let ABC (fig. 35) represent a section of the cone made by a plane through its axis. Divide the axis AD in G , so that $GD = \frac{1}{4}AD$, then G will be the centre of gravity of the cone. Join BG , then

$$BG^2 = BD^2 + DG^2.$$

But $\tan BAD = \frac{1}{2},$

therefore $BD^2 = \frac{1}{2}AD^2,$

and $DG^2 = \frac{1}{16}AD^2,$

therefore $BG^2 = \frac{9}{16}AD^2,$

and $BG = \frac{3}{4}AD,$
 $= AG;$

therefore G is the centre of the circumscribing sphere.

Hence it appears that the height of the centre of gravity of the cone will be the same in whatever position it be placed, therefore it will rest in any position.

2. A string $ABCDEP$ (fig. 37) is attached to the centre A , of a pulley whose radius is r , it then passes over a fixed point B , and under the pulley, which it touches in the points C and D ; it afterwards passes over a fixed point E , and has a weight P attached to its extremity; BE is horizontal and $= \frac{5r}{3}$, and DE is vertical: shew that if the system be in equilibrium the weight of the pulley is $\frac{5P}{2}$, and find the distance AB .

Let W be the weight of the pulley, and let θ, ϕ denote the respective inclinations of AB, BC to the horizon. The tension of the string will be throughout $= P$; hence resolving horizontally

and vertically,

$$P \cos \theta - P \cos \phi = 0 \dots\dots\dots (1),$$

$$P(1 + \sin \theta + \sin \phi) = W \dots\dots\dots (2).$$

Again,

$$AB = r \operatorname{cosec}(\theta + \phi),$$

therefore

$$EB = \frac{5r}{3} = r \{1 + \operatorname{cosec}(\theta + \phi) \cos \theta\},$$

therefore

$$\frac{2}{3} \sin(\theta + \phi) = \cos \theta \dots\dots\dots (3).$$

By (1)

$$\theta = \phi,$$

and by (3)

$$\frac{2}{3} \sin 2\theta = \cos \theta,$$

therefore

$$\sin \theta = \frac{3}{4},$$

therefore by (2)

$$W = \frac{5P}{2}.$$

Also

$$AB = r \operatorname{cosec}(\theta + \phi),$$

$$= \frac{r}{\sin 2\theta},$$

$$= \frac{r}{2 \cdot \frac{3}{4} (1 - \frac{9}{16})^{\frac{1}{2}}},$$

$$= \frac{8r}{3 \cdot 7^{\frac{1}{2}}},$$

which gives the distance AB .

DYNAMICS.

1848.

1. Two bodies acted on by gravity are projected obliquely from two given points in given directions and with given velocities: determine their position when their distance is the least possible.

Let the bodies A, B be projected from the points A, B (fig. 38) in directions AC, BC intersecting in C , and with velocities proportional to CE and CD ; upon both the bodies impress a velocity CE equal and opposite to A 's velocity, and suppose gravity not to act, the relative motion of A and B will not be affected by either of these circumstances; but A will now be reduced to rest, and B will move in a direction BG parallel to the diagonal CF of the parallelogram on CE, CD . From A draw AG perpendicular to BG , AG will be the shortest possible distance between A and B ; and A and B will be at that distance at the time (t) after the instant of projection that it takes a body animated with the velocity CF to describe the space BG , a known time therefore. Let AH and BK be the spaces due to A 's and B 's velocity of projection in time t . Through H and K draw HL and KM , each equal to the space due to gravity in the time t ; L and M are the positions required.

2. A railway train is going smoothly along a curve of 500 yards' radius at the rate of 30 miles an hour; find at what angle a plumb-line hanging in one of the carriages will be inclined to the vertical.

Let α denote the inclination of the plumb-line to the vertical,

ω the angular velocity of the train per second, r the radius of the curve.

Then the weight at the end of the plumb-line may be considered to be in equilibrium under the action of the centrifugal force, gravity, and the tension of the string.

Hence (fig. 39), by the triangle of forces,

$$\frac{\omega^2 r}{\sin(\pi - \alpha)} = \frac{g}{\sin(\frac{1}{2}\pi + \alpha)},$$

$$\text{or } \frac{\omega^2 r}{\sin \alpha} = \frac{g}{\cos \alpha},$$

$$\therefore \tan \alpha = \frac{\omega^2 r}{g}.$$

$$\text{Now } g = 32.2, \quad r = 1500, \quad \omega = \frac{30 \times 5280}{1500 \times 3600} = \frac{44}{1500};$$

$$\therefore \tan \alpha = \frac{(44)^2}{1500 \times 32.2},$$

$$= \frac{(44)^2}{48300},$$

$$= \frac{484}{12075},$$

which gives the inclination to the vertical.

3. A number of balls of given elasticity A, B, C, \dots are placed in a line; A is projected with a given velocity so as to impinge on B ; B then impinges on C , and so on: find the masses of the balls B, C, \dots , in order that each of the balls A, B, C, \dots may be at rest after impinging on the next; and find the velocity of the n^{th} ball after its impact with the $(n-1)^{\text{th}}$.

Let $m : 1$ be the ratio of the mass of n^{th} ball to that of the $(n-1)^{\text{th}}$, then the ratio of the velocity of the $(n-1)^{\text{th}}$ ball after impact to its velocity before, would be, if the balls were inelastic,

$$= \frac{1}{1+m}.$$

Since they are elastic, the ratio is

$$= 1 - (1 + e) \left(1 - \frac{1}{1 + m} \right),$$

$$= 1 - (1 + e) \frac{m}{1 + m} :$$

and since the $(n-1)^{\text{th}}$ ball is thus brought to rest, this must = 0,

$$\therefore 1 + \frac{1}{m} = 1 + e,$$

$$\text{and } m = \frac{1}{e},$$

so that the masses of the balls from a geometrical progression, whose common ratio is $\frac{1}{e}$, and the ratio of the velocity of the n^{th} ball after impact to that of the $(n-1)^{\text{th}}$ before = $\frac{1+e}{1+m} = e$; therefore if V be the initial velocity of A , velocity of n^{th} ball after impact = $e^{n-1} V$.

4. An imperfectly elastic ball is projected in a given direction within a fixed horizontal hoop, so as to go on rebounding from the surface of the hoop; find the limit to which the velocity of the ball will approach, and shew that it will attain this limit at the end of a finite time.

Let e be the modulus of elasticity, V, V_1, \dots, V_n the velocities of the ball before the first, second, ..., $(n-1)^{\text{th}}$ impacts, $\theta, \theta_1, \dots, \theta_n$ the successive angles of incidence. Then

$$V_1 \cos \theta_1 = e V \cos \theta,$$

$$V_1 \sin \theta_1 = V \sin \theta,$$

$$\therefore V_1^2 = V^2 (\sin^2 \theta + e^2 \cos^2 \theta),$$

$$= \sin^2 \theta (1 + e^2 \cot^2 \theta) V^2 :$$

$$\text{similarly } V_2^2 = \sin^2 \theta_1 (1 + e^2 \cot^2 \theta_1) V_1^2.$$

But

$$\sin^2 \theta_1 = \frac{1}{1 + \cot^2 \theta_1} = \frac{1}{1 + e^2 \cot^2 \theta},$$

therefore
$$V_2^2 = \sin^2 \theta (1 + e^2 \cot^2 \theta_1) V^2,$$

$$= \sin^2 \theta (1 + e^4 \cot^2 \theta) V^2;$$

and similarly it may be shewn that

$$V_n^2 = \sin^2 \theta (1 + e^{2n} \cot^2 \theta) V^2;$$

hence when n is indefinitely increased,

$$V_n = V \sin \theta,$$

the limit to which the velocity of the ball approaches.

Now the distances between the successive points of incidence are $2r \cos \theta_1, 2r \cos \theta_2, \dots, r$ being the radius of the circle; therefore the times of describing these spaces are

$$\begin{aligned} & \frac{2r \cos \theta_1}{V_1}, \quad \frac{2r \cos \theta_2}{V_2} \dots \dots \text{respectively,} \\ &= 2r \frac{\cot \theta_1}{(1 + \cot^2 \theta_1)^{\frac{1}{2}}} \frac{1}{V_1}, \quad 2r \frac{\cot \theta_2}{(1 + \cot^2 \theta_2)^{\frac{1}{2}}} \frac{1}{V_2} \dots \dots, \\ &= 2r \frac{e \cot \theta}{(1 + e^2 \cot^2 \theta)^{\frac{1}{2}}} \frac{(1 + e^2 \cot^2 \theta)^{\frac{1}{2}}}{\sin \theta} \frac{1}{V}, \\ & \quad 2r \frac{e^2 \cot \theta}{(1 + e^4 \cot^2 \theta)^{\frac{1}{2}}} \frac{(1 + e^4 \cot^2 \theta)^{\frac{1}{2}}}{\sin \theta} \frac{1}{V} \dots \dots, \\ &= \frac{2r}{V} e \frac{\cos \theta}{\sin^2 \theta}, \quad \frac{2r}{V} e^2 \frac{\cos \theta}{\sin^2 \theta} \dots \dots, \end{aligned}$$

therefore the ball will attain its terminal velocity, after the time

$$\begin{aligned} & \frac{2r \cos \theta}{V \sin^2 \theta} (e + e^2 + e^3 + \dots), \\ &= \frac{2r \cos \theta}{V \sin^2 \theta} \frac{e}{1 - e}. \end{aligned}$$

1849.

1. A body is projected from a given point in a horizontal direction with a given velocity, and moves upon an inclined plane passing through the point. If the inclination of the plane vary, find the locus of the directrix of the parabola which the body describes.

Let α be the inclination of the plane to the horizon; V the velocity of projection; l the latus-rectum of the parabola de-

scribed; therefore

$$l = \frac{2V^2}{g \sin \alpha},$$

$$\text{and } l \sin \alpha = \frac{2V^2}{g}.$$

But $\frac{1}{2}l \sin \alpha$ is the height of the directrix above the given point of projection; therefore this height is constant, and the locus of the directrix is a horizontal plane at a distance $\frac{V^2}{2g}$ above the given point.

2. An imperfectly elastic ball A lies on a billiard-table, determine the direction in which an equal ball B must strike it in order that they may impinge upon a side of the table at equal given angles.

The impact must be oblique and the impulse take place in the direction in which A is to go off. This direction makes with the side of the table the given angle α : let θ be the angle which B 's direction before impact makes this direction.

$V = B$'s velocity before impact,

$e =$ the modulus of elasticity.

B 's velocity $V \sin \theta$ perpendicular to the direction of the impulse will be unaltered by it: if there were no elasticity, its velocity in direction of the impulse after impact would be $\frac{1}{2}V \cos \theta$, since the balls are equal, and the impulse $\frac{1}{2}MV \cos \theta$: hence the actual impulse will be $\frac{1}{2}(1+e)MV \cos \theta$, and the actual velocity in its direction after impact $V \cos \theta - \frac{1}{2}(1+e)V \cos \theta$ or $\frac{1}{2}(1-e)V \cos \theta$.

Let $\phi =$ the angle which B 's direction after impact makes with the direction of the impulse,

$$\tan \phi = \frac{\sin \theta}{\frac{1}{2}(1-e) \cos \theta} = \frac{1}{\frac{1}{2}(1-e)} \tan \theta.$$

But $\phi = 2\alpha$,

$$\therefore \tan \theta = \frac{1}{2}(1-e) \tan 2\alpha,$$

whence θ is known.

3. A bead running upon a fine thread, the extremities of which are fixed, describes an ellipse in a plane passing through the extremities, under the action of no external force; prove that the tension of the thread for any given position of the bead is inversely proportional to the square of the conjugate diameter.

Let the bead be at the point P of the ellipse.

Since the tension of the string is the same throughout, the resultant force on the bead will bisect the angle SPH , and therefore be normal to the elliptic path. Consequently, as no force acts upon the bead in the direction of its motion, its velocity will be uniform. Now, considering the bead as moving, for the instant, in the circle of curvature at the point P , normal force $\propto \frac{\text{vel.}^2}{\text{rad. of curv.}} \propto \frac{1}{\text{rad. of curv.}}$, since the velocity is uniform: but radius of curvature $\propto CD^3$, therefore normal force $\propto \frac{1}{CD^3}$.

Now, adopting the usual notation,

tension of the string : normal force :: $PE : PF :: CD.AC : CD.PF$,
 $:: CD : BC$,

therefore tension of the string $\propto \frac{1}{CD^2}$.

4. The centres of two equal spheres (elasticity e , radius r), move in opposite directions in a circle (radius R) about a centre of force varying inversely as the square of the distance; determine the motion of the spheres after they have impinged, supposing that $e = \frac{r^2}{R^2 - r^2}$; and prove that the latus-rectum of the conic section described after the second impact will be $2e^2R$.

Let O (fig. 40) be the centre of force; C, C' the centres of the spheres. Draw OPQ perpendicular to CC' , such that CQ is perpendicular to OC , and consequently $C'Q$ to OC' . Then if CQ represent in magnitude and direction the velocity of the sphere C before impact, CP, PQ will represent its resolved parts in directions CP, PQ . Now draw QR perpendicular to $C'Q$,

meeting CP in R : the triangle QPR is evidently similar to $C'PQ$, and therefore to CPQ . Hence

$$RP : PQ :: PQ : CP,$$

$$\therefore RP : CP :: PQ^2 : CP^2 :: CP^2 : OP^2 :: r^2 : R^2 - r^2 :: e : 1.$$

Consequently RP represents in magnitude and direction the resolved part, perpendicular to OQ , of C 's velocity after impact. The velocity PQ remains unaltered by the impact; therefore the diagonal of the parallelogram PRQ drawn through P will represent in magnitude, and be parallel to the direction, of the whole velocity of C after impact. Now this diagonal makes with PQ an angle equal to RQP or $C'OP$ or COP , and is therefore parallel to OC . Hence after impact the centres of the spheres will move directly from the centre O in the lines OC , OC' . They will evidently return to the same positions C and C' , and there impinge a second time.

For the velocity of C after the second impact it is sufficient to observe that the velocity along OP will be unchanged, while that perpendicular to OP will be again diminished in the ratio of $e : 1$. Let $PR' = e.PR$. Through C draw CS equal and parallel to QR' ; join OS . Therefore the latera-recta of the first and third orbits will be to one another as $(\text{triangle } OCQ)^2 : (\text{triangle } OSC)^2$, since these triangles represent upon equal scales, half the product velocity \times perpendicular on the tangent; and we may shew that $(\text{triangle } OCQ)^2 : (\text{triangle } OSC)^2 :: 1 : e^2$; and the latus-rectum of the first or circular orbit is $2R$. Therefore that of the third is $2e^2R$.

1850.

1. Shew that it is possible to project a ball on a smooth billiard-table from a given point in an infinite number of directions, so as, after striking all the sides in order once or oftener, to hit another given point; but that this number is limited if it have to return to the point from which it was projected.

Let P (fig. 41) be the point of the table from which the ball is projected, $PQRSTU$ its course once round the table. RS

may be shewn to be parallel to QP ; and if the elasticity be perfect, equidistant with it from the line AD drawn through the corner A of the table parallel to either of them. For the angle $SRB = \text{angle } QRA = 90^\circ - RQA = 90^\circ - PQD$, therefore RS is parallel to QP . Also if QR intersect AD in F ,

$$RF : QF :: RA \sin RAF : QA \sin QAF,$$

$$:: RA \sin BRS : QA \sin DQP,$$

$$:: RA \sin ARF : QA \sin AQF.$$

But $RA \sin ARF = QA \sin AQF,$

therefore $RF = QF,$

and RS, QP are equidistant from AD .

Similarly, RS and TU are equidistant from CE . Through P draw $VDEPU$ perpendicular to the parallel lines. Then $VD = DP$ and $VE = EU$, therefore

$$PU = DE + EU - DP = DE + EV - DV = 2DE.$$

The same equation, $PU = 2DE$, holds whether P and U be on the same side of D and E or on opposite sides of either or both. Hence it is evident that by choosing the direction PQ rightly we may make the ball hit the second given point, through which the line TU will pass, after striking all the sides once: and by lessening DE or projecting the ball more nearly in the direction of the diagonal CA , we may make it strike the second point after striking all the sides twice, when PU will $= 4DE$, and so on; there being thus an infinite number of directions of projection each more nearly parallel to the diagonal CA than the preceding, which will cause the ball to hit the second given point after striking all the sides once, twice, &c., respectively.

If, however, the ball have to return to the point of projection, we must have $DE = 0$, or the direction of projection parallel to either diagonal; there being thus two directions and their opposites, or four directions in all, which will bring the ball back to its point of projection. Through this point it will pass after making each round of the table.

1851.

1. A body of given elasticity e is projected along a horizontal plane from the middle point of one of the sides of an isosceles right-angled triangle, so as, after reflexion at the hypotenuse and remaining side, to return to the same point; shew that the cotangents of the angles of reflexion are $e + 1$ and $e + 2$, respectively.

Let ABC (fig. 42) be the triangle, right-angled at A ; D the middle point of AB the point of projection: $AD = DB = a$. Let e be the modulus of elasticity. Draw DEF perpendicular to BC , making $EF = e \cdot ED$: draw FGH perpendicular to AC , making $GH = e \cdot GF$: draw HD, LF, KD : DKL will be the path of the body.

The angle of reflexion at $K = 90^\circ - LKC = \theta$ suppose,

$$\dots\dots\dots L = LHG = \phi \dots\dots\dots$$

$$\text{Now } BD = a, \therefore BE = \frac{a}{2^{\frac{1}{2}}} = \frac{1}{e} \cdot EI,$$

$$\therefore IC = 2a \cdot 2^{\frac{1}{2}} - (1 + e) BE,$$

$$= 2a \cdot 2^{\frac{1}{2}} - (1 + e) \frac{a}{2^{\frac{1}{2}}},$$

$$= (3 - e) \frac{a}{2^{\frac{1}{2}}},$$

$$\therefore CG = \frac{IC}{2^{\frac{1}{2}}} = (3 - e) \frac{1}{2} a,$$

$$\therefore AG = 2a - CG = (1 + e) \frac{1}{2} a,$$

$$\text{and } FI = ae, \quad IG = CG = (3 - e) \frac{1}{2} a,$$

$$\therefore HG = e \cdot FG = e (3 + e) \frac{1}{2} a,$$

$$\therefore \cot \phi = \frac{GH}{LG} = \frac{GH + AD}{AG} = \frac{1 + \frac{1}{2}e(3 + e) \frac{a}{2}}{\frac{1}{2}(1 + e) a},$$

$$= \frac{(2 + e)(1 + e)}{1 + e},$$

$$= 2 + e \dots\dots\dots (1).$$

Again, $\cot \theta = \tan LKC = \tan (45^\circ + KFI),$

$$\begin{aligned}
 &= \frac{1 + \tan KFI}{1 - \tan KFI}, \\
 &= \frac{1 + e \tan LHG}{1 - e \tan LHG}, \\
 &= \frac{1 + \frac{e}{2+e}}{1 - \frac{e}{2+e}} \quad \text{by (1),} \\
 &= \frac{2+2e}{2}, \\
 &= 1 + e \dots\dots\dots(2),
 \end{aligned}$$

(1) and (2) give the results required.

2. If a heavy body be projected in a direction inclined to the horizon, shew that the time of moving between two points at the extremities of a focal chord of the parabolic path is proportional to the product of the velocities of the body at the two points.

Let S (fig. 43) be the focus of the parabola, PSp the focal chord, mKM the directrix, KAS the axis; draw PN, pn perpendicular to the axis, PM, pm to the directrix. Then, since the body is acted on by no horizontal force, its horizontal velocity will be constant, and therefore the time of moving from P to p will be proportional to $PN + pn$. Also $(\text{velocity})^2$ at $P = 2g.PM$, $(\text{velocity})^2$ at $p = 2g.pm$; consequently the problem is solved if we can shew that $(PN + pn)^2 : PM.pm$ is a constant ratio.

$$\text{Now } PM = SK + SP \cos PSN = 2AS + PM \cos PSN,$$

$$\therefore PM (1 - \cos PSN) = 2AS.$$

$$\text{Similarly, } pm (1 + \cos pSn) = pm (1 + \cos PSN) = 2AS,$$

$$\therefore PM.pm \sin^2 PSN = 4AS^2 \dots\dots\dots (1).$$

$$\text{Also } PN = SP \sin PSN = PM \sin PSN = 2AS \frac{\sin PSN}{1 - \cos PSN}$$

from above.

$$\text{Similarly, } pn = 2AS \frac{\sin pSn}{1 + \cos PSN} = 2AS \frac{\sin PSN}{1 + \cos PSN};$$

$$\therefore PN + pn = 4AS \frac{\sin PSN}{1 - \cos^2 PSN} = 4AS \frac{1}{\sin PSN};$$

$$\therefore (PN + pn)^2 \sin^2 PSN = 16AS^2 \dots\dots\dots (2).$$

From (1) and (2),

$$(PN + pn)^2 : PM.pn :: 4 : 1, \text{ a constant ratio. } \quad \text{Q. E. D.}$$

NEWTON.

1849.

1. The circle described through two points of an equiangular spiral and the point of intersection of the tangents at those points will pass through the pole. Prove this, and apply the proposition to shew that the curvature at any point of an equiangular spiral varies inversely as the distance of the point from the pole.

In the equiangular spiral the tangent is inclined at a constant angle to the radius vector; hence in (fig. 44) if P_1T , P_2T be the tangents at the points P_1P_2 , S the pole of the equiangular spiral P_1P_2 ,

$$SP_1T + SP_2T = \pi,$$

and a circle can be described about the quadrilateral SP_1TP_2 , or a circle passing through P_1 , T , P_2 , will also pass the pole S .

Suppose P_1P_2 to be indefinitely near to each other, then P_1T ultimately becomes equal to P_2T , since the triangles SP_1T , STP_2 ultimately become similar and equal. Produce SP_1 to meet P_2T in R , and draw P_1R' perpendicular P_2T ; then, ultimately,

$$P_1R' = P_1R \sin SP_1T.$$

Now diameter of curvature at P_1

$$\begin{aligned} &= \text{limit } \frac{P_1P_2^2}{P_1R'} = \frac{1}{\sin SP_1T} \text{ limit } \frac{2RT.RP_2}{P_1R}, \\ &= \frac{2}{\sin SP_1T} \text{ limit } \frac{RP_1.RS}{P_1R} \text{ by the above property,} \\ &= \frac{2P_1S}{\sin SP_1T}, \end{aligned}$$

or the curvature at P_1 varies inversely as SP_1 , since SP_1T is a constant angle.

1850.

1. If any number of particles be moving in an ellipse about a force in the centre, and the force suddenly cease to act, shew that after the lapse of $\left(\frac{1}{2\pi}\right)^{\text{th}}$ part of the period of a complete revolution, all the particles will be in a similar, concentric, and similarly situated ellipse.

The velocity at any point P (fig. 45) of the orbit $= \mu^{\frac{1}{2}}CD$, and the time of revolution $\frac{2\pi}{\mu^{\frac{1}{2}}}$; therefore after the $\left(\frac{1}{2\pi}\right)^{\text{th}}$ part of a revolution, each particle will have described a space PP' equal and parallel to CD . If therefore we complete the parallelogram PCD , P' will be its angular point.

Join CP' meeting the ellipse in Q , and PD in V . Then, by a known property of the ellipse,

$$CV.CP' = CQ^2,$$

$$\text{and } CP' = 2CV;$$

$$\therefore CP'^2 = 2CQ^2,$$

$$\text{and } CP' = 2^{\frac{1}{2}}CQ;$$

therefore all the particles are in a concentric, similar, and similarly situated ellipse.

2. Two perfectly elastic balls are moving in concentric circular tubes in opposite directions and with velocities proportional to the radii: at an instant when they are in the same diameter and on opposite sides of the centre, the tubes are removed and the balls move in ellipses under the action of a force of attraction in the common centre of the circles varying inversely as the square of the distance. After one has performed in its orbit a complete revolution and the other a revolution and a half, a direct collision takes place between the balls and they interchange orbits. Find the relation between the radii of the circles and between the masses of the balls.

Let r_1, r_2 be the radii of the circles. Then the greatest and least distances in the two orbits will be r_1, r_2 in the first and r_2, r_3 in the second, where r_3 has to be determined.

Now let $h_1 h_2$ be the values of h in the two orbits, therefore

$$\begin{aligned}\frac{h_1}{h_2} &= \frac{\text{vel. in circle rad. } r_1 \times r_1}{\dots\dots\dots r_2 \times r_2}, \\ &= \frac{r_1^2}{r_2^2} \dots\dots\dots (1),\end{aligned}$$

since the velocities are proportional to the radii.

$$\text{Also } \frac{h_1^2}{h_2^2} = \frac{\text{latus-rectum in first orbit}}{\dots\dots\dots \text{second} \dots},$$

$$\begin{aligned}& \frac{r_1 r_2}{r_1 + r_2} = \frac{r_1}{r_3} \cdot \frac{r_2 + r_3}{r_1 + r_2}, \\ \therefore \frac{r_1^4}{r_2^4} &= \frac{r_1}{r_3} \cdot \frac{r_2 + r_3}{r_1 + r_2} \dots\dots\dots (2).\end{aligned}$$

Also the periodic time in the first orbit $= \frac{3}{2}$ that in the second, or

$$\begin{aligned}\frac{(r_1 + r_2)^{\frac{3}{2}}}{(r_2 + r_3)^{\frac{3}{2}}} &= \left(\frac{3}{2}\right), \\ \text{or } \frac{r_1 + r_2}{r_2 + r_3} &= \left(\frac{3}{2}\right)^{\frac{2}{3}} \dots\dots\dots (3); \end{aligned}$$

therefore, from (2) and (3),

$$\begin{aligned}\frac{r_1^4}{r_2^4} &= \left(\frac{3}{2}\right)^{\frac{2}{3}} \frac{r_1}{r_3}, \\ \text{or } \frac{r_3}{r_2} &= \left(\frac{3}{2}\right)^{\frac{2}{3}} \frac{r_2^3}{r_1^3},\end{aligned}$$

and, from (3),

$$\begin{aligned}\frac{r_3}{r_2} &= \left(\frac{2}{3}\right)^{\frac{2}{3}} \left(\frac{r_1}{r_2} + 1\right) - 1, \\ \therefore \left(\frac{3}{2}\right)^{\frac{2}{3}} \frac{r_2^3}{r_1^3} &= \left(\frac{2}{3}\right)^{\frac{2}{3}} \left(\frac{r_1}{r_2} + 1\right) - 1, \\ \text{or } \left(\frac{r_2}{r_1}\right)^4 &+ \left(\frac{2}{3}\right)^{\frac{2}{3}} \left\{1 - \left(\frac{2}{3}\right)^{\frac{2}{3}}\right\} \frac{r_2^2}{r_1} - \left(\frac{2}{3}\right)^{\frac{2}{3}} = 0,\end{aligned}$$

the equation for finding $\frac{r_2}{r_1}$. This equation has only one positive root, and that less than 1, as it ought to be, n suppose.

To find the relation between the masses m_1 and m_2 of the balls in the greater and less orbit respectively. Let v_1 and v_2 be their velocities before impact; their velocities after impact will be v_2 and v_1 respectively, m_1 and m_2 both moving after impact in the same direction as m_1 the greater did before impact. Hence, since the elasticity is perfect, momentum lost by m_1 = the whole momentum lost and gained by m_2 , or

$$m_1 (v_1 - v_2) = m_2 (v_1 + v_2);$$

and since the balls are at the same distance from the centre of force, and moving in opposite directions,

$$\frac{v_1}{v_2} = \frac{h_1}{h_2} = \frac{r_1^2}{r_2^2} \text{ by (1),}$$

$$= \frac{1}{n^2};$$

$$\begin{aligned} \therefore m_1 &= \frac{\frac{v_1}{v_2} + 1}{\frac{v_1}{v_2} - 1} m_2 \\ &= \frac{1 + n^2}{1 - n^2} m_2 \end{aligned}$$

the required relation between the masses.

1851.

1. If a body describe an ellipse round a centre of force in the focus, shew that the sum of the reciprocals of the squares of the velocities at the extremities of any chord passing through the other focus is constant.

Let PHp (fig. 46) be the chord through H . Draw the perpendiculars SY , Sy , HZ , Hz , to the tangents at those points: join SP , Sp .

Then, by a known property,

$$\frac{1}{HP} + \frac{1}{Hp} = \frac{4}{L} \quad (L \text{ the latus-rectum}),$$

$$\therefore \frac{2AC}{HP} - 1 + \frac{2AC}{Hp} - 1 = \frac{8AC}{L} - 2,$$

$$\text{or } \frac{SP}{HP} + \frac{Sp}{Hp} = \frac{8AC}{L} - 2,$$

$$\text{or } \frac{SY}{HZ} + \frac{Sy}{Hz} = \frac{8AC}{L} - 2;$$

$$\text{or } \therefore SY.HZ = Sy.Hz = BC^2,$$

$$\frac{SY^2}{BC^2} + \frac{Sy^2}{BC^2} = \frac{8AC}{L} - 2,$$

$$\text{or } SY^2 + Sy^2 \text{ is constant,}$$

and the velocities at P, p , are inversely proportional to SY, Sy ; therefore sum of the squares of the reciprocals of the velocities at Pp are constant.

COR. It may also be shewn that the sum of the squares of the velocities at the extremities of any chord passing through the centre of force is constant.

For we have shewn that

$$\frac{SY}{HZ} + \frac{Sy}{Hz} = \text{constant},$$

$$\therefore \frac{SY.HZ}{HZ^2} + \frac{Sy.Hz}{Hz^2} = \text{constant},$$

$$\text{or } BC^2 \left(\frac{1}{HZ^2} + \frac{1}{Hz^2} \right) = \text{constant},$$

$$\therefore \frac{1}{HZ^2} + \frac{1}{Hz^2} = \text{constant},$$

or, taking H as the centre of force, the sum of the squares of the velocities at the extremities of any chord passing through the centre of force is constant.*

* This corollary was set as a problem in 1848.

HYDROSTATICS.

1848.

1. An inverted vessel formed of a substance which is heavier than water contains enough of air to make it float: prove that if it be pushed down through a certain space, it will be in a position of unstable equilibrium; and determine the space in question.

When the vessel is floating partly immersed, the weight of the water displaced is equal to the weight of the vessel and of the air it contains. If the vessel be now pushed down, the water displaced, and therefore the upward pressure on the vessel, will be increased till the vessel is wholly immersed; as the vessel is now pushed down further the water displaced becomes less on account of the compression of the air in the vessel, till it comes into such a position that the weight of the water displaced is only equal to the weight of the vessel and the air it contains. This will be a position of equilibrium; and the equilibrium will be unstable, for accordingly as it is a little above or a little below this position, the weight of the water displaced will be greater or less than that of the vessel and the air it contains.

This explanation applies to a vessel of a cylindrical form; if, however, it is smaller at the top than the bottom it may come into the position of unstable equilibrium before it is wholly immersed. To find how far the vessel must be displaced so as to come into this position.

Let W be the weight of the vessel, V its volume; a and x the altitude of the column of air in the vessel in the positions

of stable and unstable equilibrium, $V''V'''$ its volumes in those positions, y the depth of its lower surface below the surface of the water in the latter position: σ the density of the water, ρ, ρ' those of the air in the two positions, p, p' its pressures in the same positions.

Then, equating the weight of the fluid displaced and of the vessel and air contained,

$$gV\sigma = W + gV'''\rho' \dots\dots\dots (1).$$

Also equating the upward pressure of the water and downward pressure of the air at their common surface,

$$g\sigma y = p' \dots\dots\dots (2).$$

Also, since pressure and density vary inversely as volume,

$$\frac{p'}{p} = \frac{\rho'}{\rho} = \frac{V}{V'''} \dots\dots\dots (3).$$

Again, when the form of the vessel is known, V'' and V''' will be known in terms of a and x . Hence equations (1), (2), and (3), will be sufficient for the determination of x and y , as well as the other unknown quantities, viz. p', ρ' , and V'' . Hence $y - x$, or the depth of upper surface of the air below that of the water, is known, and added to the difference of the heights of the same two surfaces in the original position of equilibrium, gives the space through which the vessel must be depressed.

2. A uniform piston, terminated by a plane of area A , perpendicular to its side, is inserted into an orifice in a vessel containing fluid; prove that the work done in gently pushing in the piston through a small space s is ultimately equal to the work done in lifting a portion of the fluid of volume As through a height equal to the depth of the centre of gravity of the plane below the surface of the fluid.

If the space s be indefinitely small, the pressure on each element of the piston will be unaltered by the change of the piston's position.

Hence, if s be indefinitely small,

work done = product of whole pressure on area $A \times s$, the space through which it is moved perpendicular to itself,
 = pressure at depth \bar{z} of the centre of gravity of $A \times$ area $A \times s$,
 = $g\rho\bar{z}As$,
 = $g\rho As.\bar{z}$,
 = weight of volume As of the fluid $\times \bar{z}$,
 = work done in raising the volume As through the space \bar{z} .

3. Two equal slender rods AB , AC , moveable about a hinge at A , and connected by a string BC , rest with the angle A immersed in a given fluid; determine the tension of the string BC .

Let T = tension of the string,

w = weight of each rod,

$2a$ = its length,

$2l$ = the length of the part immersed,

α = its inclination to the horizon.

Then the rod is kept at rest by its weight, the tension of the string, the action at the hinge and the fluid pressures which have for resultant a vertical upward pressure w acting at a distance l from A .

Hence, taking moment about A ,

$$w.a \cos \alpha - w.l \cos \alpha - T.2a \sin \alpha = 0,$$

$$\therefore T = \frac{a-l}{2a} \cot \alpha.w$$

is the required tension.

1849.

A body floats in a mixture of two given fluids with a volume A immersed; one half of the mixture being removed, and its place supplied by an equal quantity of the lighter fluid, the same body floats with a volume $A + B$ immersed. Determine

the ratio of the quantities of fluid in the original mixture, supposing the volume of the mixture to be equal to the sum of the volumes of the component fluids.

Explain the result when the densities of the fluids are as $A + B$ to $A - B$.

Let V, V' be the original volumes of the fluids; σ, σ' their specific gravities. Their volumes in the second mixture will be $\frac{1}{2}V$ and $\frac{1}{2}V' + \frac{1}{2}(V' + V)$ or $V' + \frac{1}{2}V$; the specific gravities of the mixtures will be

$$\frac{V\sigma + V'\sigma'}{V + V'} \text{ and } \frac{\frac{1}{2}V\sigma + (V' + \frac{1}{2}V)\sigma'}{V' + V'};$$

hence, if W be the weight of the body,

$$W = A \frac{V\sigma + V'\sigma'}{V + V'},$$

$$\text{also} = (A + B) \frac{\frac{1}{2}V\sigma + (V' + \frac{1}{2}V)\sigma'}{V' + V'};$$

$$\therefore A(V\sigma + V'\sigma') = (A + B) \left\{ \frac{1}{2}V\sigma + (V' + \frac{1}{2}V)\sigma' \right\},$$

$$\therefore \left\{ A\sigma - \frac{1}{2}(A + B)\sigma - \frac{1}{2}(A + B)\sigma' \right\} V = B\sigma' V',$$

$$\text{or } \frac{V}{V'} = \frac{2B\sigma'}{(A - B)\sigma - (A + B)\sigma'},$$

the required ratio.

If the densities, and therefore the specific gravities, are as $A + B$ to $A - B$, $V' = 0$, shewing that the fluid cannot be a mixture of fluids of different specific gravities; in fact, the conditions of the problem then become impossible.

1850.

1. A conical vessel containing a given quantity of fluid has its axis vertical, and another cone with the same vertical angle is placed to float in the fluid with its vertex downwards; find how much the fluid will rise in consequence.

Let h be the depth of the original cone of fluid, k the depth to which the vertex of the floating cone will sink; k is known from the specific gravities of the fluid and floating cone. z the

height through which the fluid will rise. Then

$$\frac{\text{volume of the cone height } h+z}{\dots\dots\dots h} = \frac{(h+z)^3}{h^3},$$

$$\text{and } \frac{\text{volume of the cone height } k}{\dots\dots\dots h} = \frac{k^3}{h^3}.$$

Therefore, subtracting,

$$\frac{\text{volume of the fluid after the cone is put into it}}{\text{its original volume}} = \frac{(h+z)^3 - k^3}{h^3},$$

$$= 1,$$

since the quantity of fluid is unaltered; therefore

$$(h+z)^3 - k^3 = h^3$$

$$\text{and } z = (h^3 + k^3)^{\frac{1}{3}} - h$$

is the required space.

2. A hollow cylinder containing air is fitted with an airtight piston which, when the cylinder is placed vertically, is at a given height above the base; the cylinder being now inverted and placed vertically in a fluid, sinks partly below the surface; find the position of equilibrium.

Let p be the pressure of the air in the cylinder before the cylinder is inverted, and which is known from the given height (h) of the piston above the base: p is the pressure due to the weight of the piston and atmosphere, Π the atmospheric pressure, w the weight of the cylinder and piston, z the depth below the surface of the fluid of the piston in the position of equilibrium, y the distance of the piston from the base of the cylinder, ρ , ρ' the densities of the fluid and uncompressed air; then, for the equilibrium of the piston,

fluid pressure from beneath = pressure due to the weight of the piston + the pressure of the air in the inverted cylinder,

$$\text{or } g\rho z + \Pi = p - \Pi + \frac{h}{y} \rho \dots\dots\dots (1).$$

Also w + weight of the air in the cylinder

= weight of the fluid displaced,

$$\therefore w + g\rho'V = g\rho \frac{z}{h} V \dots\dots\dots (2),$$

where V is the volume and h' the height of the cylinder.

From equation (2) z is known, and thence y from (1).

1851.

1. A hollow cone floats in a fluid with its vertex upwards and axis vertical; determine the density of the air in the hollow cone.

Let p be the pressure of the air in the cone, Π that of the atmosphere, w the weight of the cone, h its height, z the height of the cone of compressed air, y the depth of its base below the surface of the fluid, ρ, ρ' the densities of the fluid and uncompresssed air.

Then, equating the pressures at the common surface of the air and fluid,

$g\rho y + \Pi$ = pressure of the compressed air,

$$= p = \frac{h}{z} \Pi \dots\dots\dots (1).$$

Also w + weight of the air in the cylinder

= weight of the fluid displaced,

$$\text{or } w + g\rho'V = g\rho \frac{z^3 - (z-y)^3}{h^3} V \dots\dots\dots (2),$$

where V is the volume of the cone.

From equations (1) and (2) z and y are known, and thence p , and the required density.

OPTICS.

1848.

1. If Q, q (fig. 47) be two points in the radius of a spherical reflecting surface whose centre is E , such that $EQ : Eq :: \sin$ of the angle of incidence : \sin of the angle of refraction, determine geometrically the position of the point P , so that a ray proceeding from Q and incident upon the surface at P may after refraction proceed from q .

Bisect Qq in m , and through m draw mP perpendicular to EQ meeting the circle in P ; P will be the point required. For if we join PE, Pq, PQ , we have

$$\sin EPQ : \sin PQE :: QE : PE,$$

$$\text{and } \sin EPq : \sin PqE :: qE : PE.$$

$$\text{Now } \sin PQE = \sin PqE, \quad \therefore \angle PQE = \angle PqQ,$$

$$\therefore \sin EPQ : \sin EPq :: EQ : Eq :: \mu : 1$$

by the question, therefore the ray QP after refraction at P will proceed as if from q .

2. If a ray of light, after being reflected any number of times in one plane, at any number of plane surfaces, return on its former course, prove that the same will be true of any ray parallel to the former which is reflected at the same surfaces in the same order, provided the number of reflections be even.

Let $PQRS$ (fig. 48) be the course of any ray which starting from P , after reflection at Q, R and S , arrives again at P , and is there reflected in the direction PQ of the original propagation. Let $P'Q'R'S'$ be the course of another ray starting from P' in a direction $P'Q'$ parallel to PQ ; we have to shew that after reflection at S' , this ray will proceed to P' , and there be reflected in the direction $P'Q'$. Join $S'P'$.

Then since the angle $Q'P'A = QPA$, and $QPA = SPP'$, therefore the triangle $IP'P'$ is isosceles, and the perpendicular from I'

on $P'Q' =$ that from P' on PS . Similarly the perpendicular from Q' on $QR =$ that from Q on $Q'P' =$ that from P on $P'Q'$ since PQ is parallel to $P'Q'$; therefore the perpendicular from Q' on $QR =$ that from P' on PS . By similar reasoning it may be shewn that the perpendicular from S' on $PS =$ that from R' on $QR =$ that from Q' on QR since $Q'R'$ is parallel to $QR =$ that from P' on PS . Hence $P'S'$ is parallel to PS ; therefore $S'P'$ is the direction in which $R'S'$ will be reflected from S' , and $P'Q'$ is that in which $S'P'$ will be reflected from P' .

The same proof may be extended to any even number of reflections. If the number of reflections were not even we might still shew that P', S' were equidistant from PS , but they would be on opposite sides of it, as P', R' are of PR , and the propositions would not be true in that case.

1849.

If the angle of a hollow cone, polished internally, be any submultiple of 180° , a cylindrical pencil of rays incident parallel to the axis will, after a certain number of reflections, be a cylindrical pencil parallel to the axis, and of the same diameter as the incident pencil.

Let fig. 49 represent a section of the cone and the light by a plane through the axis CD of the cone, and let $m_1 m_2 \dots$ be the successive points when the ray $PQ_1 Q_2 Q_3 \dots$ cuts the axis CD .

(1). Let the angle ACB be an even submultiple of $180^\circ = \frac{180^\circ}{2n}$ suppose, or $\frac{90^\circ}{n}$.

$$\text{Now the angle } Q_1 m_1 D = ACD + CQ_1 m_1 = ACD + AQ_1 P, \\ = 2ACD = ACB,$$

$$\text{and the angle } Q_2 m_2 D = BCD + CQ_2 m_2 = BCD + Q_1 Q_2 B, \\ = BCD + BCD + Q_1 m_1 D, \\ = 2ACB;$$

$$\text{similarly } Q_3 m_3 D = 3ACB,$$

$$\dots\dots = \dots\dots,$$

$$\text{and } Q_n m_n D = nACB = 90^\circ,$$

or after the n^{th} reflection the ray will be perpendicular to the axis CD , and will proceed in a path exactly similar to that already described, finally emerging in a direction parallel to Q_1P , and at the same distance as Q_1P from the axis CD , but on the opposite side of it.

(2). Let the angle ACB be an odd submultiple of $180^\circ = \frac{180^\circ}{2n+1}$ suppose.

$$\begin{aligned}\text{Now the angle } Q_1Q_2B &= BCD + Q_1m_1D, \\ &= \frac{1}{2}ACB + ACB \text{ by the above,} \\ &= \frac{3}{2}ACB,\end{aligned}$$

$$\begin{aligned}\text{and the angle } Q_2Q_1A &= ACD + Q_2m_2D, \\ &= \frac{1}{2}ACB + 2ACB, \\ &= \frac{5}{2}ACB;\end{aligned}$$

$$\text{similarly } Q_3Q_4B = \frac{7}{2}ACB,$$

$$\text{and } Q_nQ_{n+1}A = \frac{2n+1}{2}ACB = 90^\circ;$$

or after n reflection the ray will be perpendicular to the side CA or CB , at which it has next to be reflected, and will therefore after that reflection return by the same path as it came by, and will emerge in the direction Q_1P .

Hence, whether ACB be an even or odd submultiple of 180° , the emergent rays will form a cylinder equal in diameter to the cylinder of incident rays, and having its axis coincident with the axis of that cylinder, if the angle ACB be an odd submultiple of 180° ; or if the angle ACB be an even submultiple of 180° , the axes of the emergent and incident pencils will lie in the same plane with the axis of the cone at equal distances on opposite sides of it.

1850.

1. If a luminous point be seen after reflection at a plane mirror by an eye in a given position, there is a certain space within which the image of the point can never be situated, however the position of the plane mirror be changed: find this space.

It is easily seen that the distance from the eye of the image formed by the mirror equals the actual length of the ray by which the point is seen. This can never be less than the direct distance of the point from the eye; hence the image can never be situated within the sphere which has the direct distance between the point and the eye for radius.

2. If α be the angle which every diameter of a circular disc subtends at a luminous point, shew that the ratio of the light which falls on the disc to the whole light emitted is as $\sin^2 \frac{1}{4} \alpha : 1$.

About the luminous point as centre describe a sphere with radius unity; also with the luminous point for vertex and the circular disc as base describe a right cone. Then the light received on the circular disc : whole light emitted :: the portion of the surface of the sphere intercepted by the cone : whole surface of the sphere.

Now by a known property of the sphere, the surface of any portion of the sphere cut off by any plane is proportional to the difference of the radius of the sphere and the distance of the cutting plane from the centre. Hence the surface intercepted by the above cone : whole surface of the sphere :: $1 - \cos \frac{1}{2} \alpha : 2 :: \sin^2 \frac{1}{4} \alpha : 1$, which is therefore the ratio of the light received on the circular disc to the whole light emitted.

1851.

A sphere composed of two hemispheres of different refractive powers is placed in the path of a pencil of light in such a manner that the axis of the pencil is perpendicular to the plane of junction and passes through the centre: determine the geometrical focus of the refracted pencil.

Let r be the radius of the sphere, u the distance of the focus of incident rays from the centre; $v_1 v_2 v_3$ the distances of the geometrical foci after the successive refractions, positive lines being measured in the direction opposite to that of the incident light; $\mu_1 \mu_2$ the refractive indices of the two hemispheres.

$$\text{Then } \frac{1}{v_1} = -\frac{\mu_1 - 1}{r} + \frac{\mu_1}{u} \dots \dots \dots (1),$$

$$\frac{1}{v_2} = \frac{\mu_1}{\mu_2} \frac{1}{v_1} \dots \dots \dots (2),$$

$$\frac{1}{v_3} = \frac{\frac{1}{v_2} - 1}{r} + \frac{1}{\mu_2} \frac{1}{v_2} \dots \dots \dots (3),$$

$$(3) + (2) \times \frac{1}{\mu_2} + (1) \times \frac{\mu_1}{\mu_2},$$

$$\frac{1}{v_3} = -\frac{1}{\mu_2} \{ \mu_1 (\mu_1 - 1) + \mu_2 (\mu_2 - 1) \} \frac{1}{r} + \frac{\mu_1^2}{\mu_2} \frac{1}{u},$$

which gives v_3 the distance from the centre of the sphere of the geometrical focus after refraction.

ASTRONOMY.

1849.

1. There are two walls of equal known height at right angles to each other, and running in known directions; shew how to find the sun's altitude and azimuth by observing the breadth of the shadows of the two walls at any given time. And prove that the sum of the squares of the breadths of the shadows will be the same whatever be the direction of the walls.

Let a , b be the observed breadths of the shadows, h the known height of the walls; θ the angle between the base of the wall, the breadth of whose shadow is a , and the line joining the shadow of the top of the line of intersection of the walls with the bottom of that line, ϕ the sun's altitude. Then

$$\theta = \tan^{-1} \frac{a}{b}, \quad \text{and} \quad \phi = \tan^{-1} \frac{h}{(a^2 + b^2)^{\frac{1}{2}}}$$

are known. Let α be the angle between the wall whose breadth is a and the plane of the meridian; then $\alpha \sim \theta$ is the angle between the plane of the meridian and the vertical plane through the sun, or the sun's azimuth. Hence both the altitude and azimuth are known.

Also $a^2 + b^2 =$ the square of the length of the shadow of the line of intersection of the walls; and the height of this line is the same whatever be the direction of the walls, or $a^2 + b^2$ is independent of that direction.

2. If the same two stars rise together at two places, the places will have the same latitude. And if they rise together at one place and set together at the other, the places will have equal latitudes, but one north and the other south.

From the bisection of SS' the great circle passing through the two stars S , S' draw a quadrant of a great circle perpendicular to SS' towards the north pole terminating in the

point T , and another towards the south pole terminating in T' . First, suppose the great circle containing these quadrants to have its point which is nearest to P the north pole, to the left of P (drawing the stars on the convex part of the sphere). Then, in order that the stars S, S' may rise together at any place, its zenith must at some time in the 24 hours come to T ; its co-latitude therefore must be TP . Hence if S, S' rise together at two places, the co-latitude of each must be TP ; hence the latitudes of the places are the same. If the point of the circle nearest to P lie to the right of P , the zenith of the place must pass through T' in their daily path and therefore have the same latitude, viz. $90^\circ - T'P$, S, P' being the south pole.

If the same stars rise together at one place and set together at another, the zenith of one must pass through T and that of the other through T' in their daily paths; hence they will still have equal latitudes, but one will be north and the other south.

1850.

1. Prove that all stars which rise at the same instant at a place within certain limits of latitude, will, after a certain interval, lie in a vertical great circle; and determine those limits.

This will happen when the zenith of the place comes to that great circle of the heavens which at the time of the stars' rising was the horizon of the place. Hence it can only happen for those places for which the altitude of the pole is less than the co-latitude: but the altitude of the pole is the latitude, hence if l be the latitude, l must be less than $90^\circ - l$ or l less than 45° .

2. Shew how to find the days of the year on which the light of the sun reflected by a given window which has a south aspect will be thrown into some one of the lower windows of an opposite range of buildings.

Corresponding to each window opposite, let that point of the heavens be determined, which lies in the same plane as that window and the horizontal line through the reflecting window pointing to the south, and at the same angular distance from this

line as the opposite window in question, and on the opposite side of it. Let the north polar distance of this point or its declination be then observed; the reflected light will enter the window in question on those days when the sun has this declination. Similarly, the days when the reflected light will enter the other windows may be determined.

1851.

Altitudes of the same heavenly body are observed from the deck of a ship and from the top of the mast the height of which from the deck is known: find the dip of the horizon and the true altitude.

Let $AB = x$, $BC = h$ (fig. 50) be the height of the deck from the sea, and of the mast respectively; $OA = r$ the radius of the earth. The difference (α) of the observed altitudes is the angle BFC or DOE .

$$\text{Now } \cos COE = \frac{r}{r+h+x}, \text{ and } \cos COD = \frac{r}{r+x},$$

$$\therefore DOE = \cos^{-1} \frac{r}{r+h+x} - \cos^{-1} \frac{r}{r+x} = \alpha,$$

an equation for the determination of x .

Then the dip of the horizon $= OBD = \sin^{-1} \frac{r}{r+x}$ is known, and subtracted from the altitude observed at B gives the true altitude.

From the above equation we may determine x with sufficient accuracy thus:

$$\begin{aligned} \frac{r}{r+x} \left\{ 1 - \frac{r^2}{(r+h+x)^2} \right\}^{\frac{1}{2}} - \frac{r}{r+h+x} \left\{ 1 - \frac{r^2}{(r+x)^2} \right\}^{\frac{1}{2}} &= \sin \alpha, \\ \text{or } \frac{\{2r(h+x)\}^{\frac{1}{2}}}{r} - \frac{(2rx)^{\frac{1}{2}}}{r} &= \sin \alpha, \end{aligned}$$

omitting h and x in comparison of r ;

$$\therefore \frac{2(h+x)}{r} = \sin^2 \alpha - 2 \sin \alpha \left(\frac{2x}{r} \right)^{\frac{1}{2}} + \frac{2x}{r},$$

$$\therefore \left(\frac{2x}{r} \right)^{\frac{1}{2}} = \frac{1}{2} \sin \alpha - \frac{h}{r} \operatorname{cosec} \alpha,$$

$$\text{or } x = \left(\frac{1}{2} \sin \alpha - \frac{h}{r} \operatorname{cosec} \alpha \right)^2 \frac{r}{2}.$$

PART II.

PART II.

EUCLID.

1848.

1. AB, CD, (fig. 51) are any two chords of a circle passing through a fixed point O, EF any chord parallel to AB; join CE, DF meeting AB in the points G and H, and DE, CF meeting AB in the points K and L: shew that the rectangle $OG.GH = OK.OL$.

The triangles OCG, OHD have the common angle O, and

$$\begin{aligned}\angle OCD &= 180^\circ - \angle EFD = \angle EFH, \\ &= \angle OHD,\end{aligned}$$

since EF is parallel to AB; hence the triangles OCG, OHD are similar, therefore

$$\begin{aligned}OC : OG &:: OH : OD, \\ \text{or } OG.OH &= OC.OD.\end{aligned}$$

Again, $\angle OCD = \angle OEF = \angle DKO$,

since EF is parallel to AB; hence the triangles OLC, ODK are similar, therefore

$$\begin{aligned}OC : OL &:: OK : OD, \\ \text{or } OL.OK &= OC.OD, \\ \therefore OG.OH &= OL.OK.\end{aligned}$$

2. In a given circle inscribe a rectangle equal to a given rectilineal figure.

Let AB (fig. 52) be a diameter of the given circle ABC. Draw a square which shall be equal to the rectilineal figure

(*Euc.* II. 14); through D any point of AB draw DE perpendicular to AB, a third proportional to AB and the side of the square. Through E draw EC parallel to AB, meeting the circle in C; join AC, BC, and complete the parallelogram ACBF: it shall be the rectangle required.

For C and F are each right angles, being the angles in a semicircle, therefore ACBF is a rectangle. Also its area equals AB.DE which equals, by construction, the square which equals the given rectilineal figure; therefore it is the rectangle required.

3. Through a given point A (fig. 53) describe a circle which shall touch a given circle BCD, and intersect another given circle LEF in a chord passing through a given point G.

From G draw any line GEF intersecting the circle LEF in the points E, F; join GA, and in GA produced if necessary, take the point H, such that $GA.GH = GE.GF$. Through the points A, H describe any circle cutting the circle BCD in the points B, C; join BC, and produce it to meet GA in K. From K draw KD a tangent to the circle BCD. About the triangle AHD describe a circle, it shall be the circle required.

And first it shall touch the circle BCD: for since KD touches the circle BCD,

$$KD^2 = KB.KC = KA.KH,$$

since one circle has been made to pass through A, H, C, and B; and therefore KD touches the circle in question as well as the circle BCD, therefore the two circles touch. Also the chord in which it intersects the circle LEF will pass through G. For suppose L to be one of the points in which it intersects the circle LEF; join GL and produce it to meet the two circles in M, M'. Then

$$GL.GM = GE.GF = GA.GH, \text{ by construction;}$$

$$\text{also } GL.GM' = GA.GH,$$

since H, A, L, M', lie in the circumference of the same circle, therefore $GM = GM'$ or the points M and M' coincide, and GLM is the chord in which the circles intersect, and the chord passes through G as required.

Hence the circle drawn as above described fulfils the required conditions, and is therefore the circle sought.

1849.

Three circles are described, each of which touches one side of a triangle ABC (fig. 54), and the other two sides produced. If D be the point of contact of the side BC, E that of CA, and F that of AB, shew that $AE = BD$, $BF = CE$, and $CD = AF$.

Let AB, AC touch the circle GDH in G, H; then

$$BG = BD, \text{ and } CH = CD,$$

$$\text{also } AG = AH,$$

$$\therefore AB + BD = AC + CD = \text{semiperimeter of the triangle,}$$

similarly, $BA + AE = \text{semiperimeter of the triangle} :$

$$\therefore BA + AE = AB + BD,$$

$$\text{and } AE = BD;$$

and similarly, $BF = CE$ and $CD = AF$.

1851.

1. Let T (fig. 55) be a point without a circle, whose centre is C; from T draw two tangents TP, TQ; also through T draw any line meeting the circle in V, and PQ in R, and draw CS perpendicular to TV; then $SR.ST = SV^2$.

Join CT, intersecting PQ in U at right angles; draw CP; it will be perpendicular to PT.

Since the triangles CTS, RTU are similar,

$$CT : TS :: RT : TU,$$

$$\therefore TS.TR = TC.TU,$$

$$\begin{aligned} \text{or } ST^2 - ST.SR &= CT^2 - CT.CU, \\ &= ST^2 + CS^2 - CT.CU, \end{aligned}$$

$$\therefore ST.SR = CT.CU - CS^2;$$

but CPT, PUC are both right angles,

$$\begin{aligned} \therefore CT.CU &= CP^2 = CV^2, \\ &= SV^2 + CS^2, \end{aligned}$$

$$\therefore ST.SR = SV^2.$$

2. If a circle be described round the point of intersection of the diameters of a parallelogram as a centre, shew that the sum of the squares of the lines drawn from any point in its circumference to the four angular points of the parallelogram is constant.

Join P any point in the circle with A, B, C, D (fig. 56) the angular points of the parallelogram, and with O the centre of the circle. Then, since $OB = OD$, the square of BP is greater than the squares of OP and OB by the rectangle by which the squares of OP and OD are greater than the square of DP (*Euc.* II. 12, 13); hence

$$BP^2 + DP^2 = OB^2 + OD^2 + 2OP^2 \text{ is constant :}$$

similarly, $AP^2 + CP^2 = AO^2 + OC^2 + 2OP^2$ is constant,

therefore also $AP^2 + BP^2 + CP^2 + DP^2$ is constant.

3. (α). Let B (fig. 57) be any point in the circumference of a circle whose centre is A; in AB take two points C and D, such that $AC.AD = AB^2$; bisect DC in E, and draw EF at right angles to AE; in EF take any point G, then will the tangent drawn from G to the circle be equal to GC.

Draw GF the tangent to the circle, join CG; then

$$AG^2 = GF^2 + AF^2 = GF^2 + AB^2,$$

$$\begin{aligned} \text{also } AG^2 &= GE^2 + AE^2 = GE^2 + CE^2 + AC.AD \text{ (} \textit{Euc. II. 6}), \\ &= CG^2 + AB^2 \text{ by construction;} \end{aligned}$$

$$\therefore GF^2 + AB^2 = CG^2 + AB^2,$$

$$\text{and } GF = CG.$$

(β). Describe a circle which shall pass through a given point, touch a given straight line, and cut orthogonally a given circle.

Let D be the given point, BFL the given circle, KH the given line intersecting AD in H.

In AD take $AC : AB :: AB : AD$, and HK a mean proportional to HD and HC; the circle through CD and K shall be the required circle.

For the centre of this circle will be at some point G of EG , and since $GF = GC$, will pass through F , cutting the circle BFL orthogonally at that point.

Also since HK is a mean proportional to HD , HC , or $HK^2 = HD.HC$, HK will touch the circle through the points C , D , K ; hence that circle fulfils all the required conditions and is the circle sought.

ALGEBRA.

1848.

1. A walks to Trumpington and back by Granchester in $1\frac{1}{4}$ hour, starting between 2 o'clock and $2\frac{3}{4}$; B walks the same distance in the same direction in $1\frac{1}{2}$ hour, starting between 2 and $2\frac{1}{4}$: find the chance that A overtakes B before he gets home.

Unless A starts after B he cannot *overtake* him.

Now if he starts between 2 and $2\frac{1}{4}$, it is an even chance whether he starts first or not; otherwise he cannot. Hence A 's chance of starting before $B = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

Again, A gets home between $3\frac{1}{4}$ and 4, B between $3\frac{1}{2}$ and $3\frac{3}{4}$; therefore A has an even chance of getting home first; therefore also his chance of getting home last $= \frac{1}{2}$, and chance of his *not* overtaking $B =$ chance of his starting first + chance of his getting home last

$$= \frac{1}{2} + \frac{1}{6} = \frac{2}{3};$$

therefore chance of A overtaking $B = 1 - \frac{2}{3},$
 $= \frac{1}{3},$

the required chance.

2. A parallelopiped is cut by three systems of parallel planes given in number, parallel to the three pairs of opposite faces respectively: find the total number of parallelopipeds formed in every way.

Let $m, n, p,$ be the given number of intersecting planes parallel to the three sides respectively; we thus have $m + 2, n + 2, p + 2$ parallel planes in three several directions.

Now, out of the first set of parallel planes we may make $\frac{(m+2)(m+1)}{2}$ sets of two each. Similarly, out of the other two sets we may make $\frac{(n+2)(n+1)}{2}, \frac{(p+2)(p+1)}{2}$ sets respectively.

Now each parallelopiped is formed by taking one out of each of the above sets of two parallel planes, therefore the total number of parallelopipeds will be

$$\frac{(m+2)(m+1)}{2} \cdot \frac{(n+2)(n+1)}{2} \cdot \frac{(p+2)(p+1)}{2},$$

$$= \frac{(m+1)(n+1)(p+1)(m+2)(n+2)(p+2)}{8},$$

the required number.

$$3. \text{ If } (\beta - \alpha)(y - m\alpha) = (n\beta - m\alpha)(x - \alpha) \dots \dots \dots (1),$$

$$\text{and } (\beta' - \alpha')(y - m\beta') = (m\beta' - n\alpha')(x - \beta') \dots \dots \dots (2),$$

$$\text{shew that } \left(\frac{1}{\alpha\alpha'} - \frac{1}{\beta\beta'} \right) x = \frac{\alpha + \alpha'}{\alpha\alpha'} - \frac{\beta + \beta'}{\beta\beta'}.$$

Taking (1) $(\beta' - \alpha') - (2) (\beta - \alpha)$, so as to eliminate y , we get
 $m(\beta - \alpha)(\beta' - \alpha')(\beta' - \alpha) = \{(n\beta - m\alpha)(\beta' - \alpha') - (m\beta' - n\alpha')(\beta - \alpha)\}x$
 $- (n\beta - m\alpha)(\beta' - \alpha')\alpha + (m\beta' - n\alpha')(\beta - \alpha)\beta',$

or $m\{(\beta - \alpha)(\beta'^2 - \alpha'\beta') + (\beta' - \alpha')(\alpha^2 - \alpha\beta)\} = (n - m)(\beta\beta' - \alpha\alpha')x$
 $- n\{(\beta' - \alpha')\alpha\beta + (\beta - \alpha)\alpha'\beta'\} + m\{(\beta' - \alpha')\alpha^2 + (\beta - \alpha)\beta'^2\},$

$$\therefore (n - m)\{(\beta' - \alpha')\alpha\beta + (\beta - \alpha)\alpha'\beta'\} = (n - m)(\beta\beta' - \alpha\alpha')x,$$

and dividing by $\alpha\alpha'\beta\beta'$,

$$x \left(\frac{1}{\alpha\alpha'} - \frac{1}{\beta\beta'} \right) = \frac{\alpha + \alpha'}{\alpha\alpha'} - \frac{\beta + \beta'}{\beta\beta'}.$$

4. (α). Shew that the integral parts of $(3^{\frac{1}{2}} + 1)^{2m+1}$ and $(3^{\frac{1}{2}} + 1)^{2m} + 1$ are respectively divisible by 2^{m+1} where m is any integer whatever.

The integral part of $(3^{\frac{1}{2}} + 1)^{2m+1}$ is $(3^{\frac{1}{2}} + 1)^{2m+1} - (3^{\frac{1}{2}} - 1)^{2m+1}$, since it is a whole number, and $(3^{\frac{1}{2}} - 1)^{2m+1}$ is less than 1.

Now generally

$$x^{2m+1} - y^{2m+1} = (x - y) \{(x^{2m} + y^{2m}) + xy(x^{2m-2} + y^{2m-2})$$

$$+ x^2y^2(x^{2m-4} + y^{2m-4}) + \dots + x^m y^m \dots \dots \dots (\Lambda),$$

$$\text{and if } x = 3^{\frac{1}{2}} + 1, \quad y = 3^{\frac{1}{2}} - 1,$$

$$x - y = 2, \quad xy = 2,$$

$$\therefore (3^{\frac{1}{2}} + 1)^{2m+1} - (3^{\frac{1}{2}} - 1)^{2m+1} = 2 [\{(3^{\frac{1}{2}} + 1)^{2m} + (3^{\frac{1}{2}} - 1)^{2m}\}$$

$$+ 2 \{(3^{\frac{1}{2}} + 1)^{2m-2} + (3^{\frac{1}{2}} - 1)^{2m-2}\} + \dots + 2^m].$$

This part of the question, then, reduces itself to shewing that $(3^{\frac{1}{2}} + 1)^{2m} + (3^{\frac{1}{2}} - 1)^{2m}$ is divisible by 2^m .

Again, the integral part of $(3^{\frac{1}{2}} + 1)^{2m} + 1$ is $(3^{\frac{1}{2}} + 1)^{2m} + (3^{\frac{1}{2}} - 1)^{2m}$, since this is a whole number and $(3^{\frac{1}{2}} - 1)^{2m}$ is less than 1; hence the second part of the question reduces itself to shewing that $(3^{\frac{1}{2}} + 1)^{2m} + (3^{\frac{1}{2}} - 1)^{2m}$ is divisible by 2^{m+1} , and therefore includes the first part.

$$\begin{aligned}\text{Now } (3^{\frac{1}{2}} + 1)^{4n} + (3^{\frac{1}{2}} - 1)^{4n} &= (4 + 2 \cdot 3^{\frac{1}{2}})^{2n} + (4 - 2 \cdot 3^{\frac{1}{2}})^{2n}, \\ &= 2^{2n} \{ (2 + 3^{\frac{1}{2}})^{2n} + (2 - 3^{\frac{1}{2}})^{2n} \},\end{aligned}$$

which is evidently divisible by 2^{2n+1} . Also generally

$$\begin{aligned}x^{4n+2} + y^{4n+2} &= (x^2 + y^2) \{ (x^{4n} + y^{4n}) - x^2 y^2 (x^{4n-4} + y^{4n-4}), \\ &\quad + x^4 y^4 (x^{4n-8} + y^{4n-8}), \\ &\quad - \dots + (-)^n x^{2n} y^{2n} \};\end{aligned}$$

$$\begin{aligned}\therefore (3^{\frac{1}{2}} + 1)^{4n+2} + (3^{\frac{1}{2}} - 1)^{4n+2} &= \{ (3^{\frac{1}{2}} + 1)^2 + (3^{\frac{1}{2}} - 1)^2 \} \\ &[\{ (3^{\frac{1}{2}} + 1)^{4n} + (3^{\frac{1}{2}} - 1)^{4n} \} - 2^2 \{ (3^{\frac{1}{2}} + 1)^{4n-4} + (3^{\frac{1}{2}} - 1)^{4n-4} \} + \dots + (-)^n 2^{2n}], \\ &= 2^3 [\dots],\end{aligned}$$

which, since $(3^{\frac{1}{2}} + 1)^{4n} + (3^{\frac{1}{2}} - 1)^{4n}$ is divisible by 2^{2n+1} , is divisible by 2^{2n+4} , and therefore *à fortiori* by 2^{2n+2} or $2^{(2n+1)+1}$.

Hence, whether m be of the form $2n$ or $2n + 1$,

$$(3^{\frac{1}{2}} + 1)^{2m} + (3^{\frac{1}{2}} - 1)^{2m}$$

is divisible by 2^{m+1} , and both parts of the proposition are true.

(β). Prove that for a given integral value of a , there are

(1), a integral values of b which will make the integral part of $(a + b^{\frac{1}{2}})^{2m+1}$ divisible by 2^{m+1} ;

(2), a integral values of b which will make the integral part of $(a + b^{\frac{1}{2}})^{2m+1} + 1$ divisible by 2^{m+1} ;

(3), $2a$ integral values of b which will make the integral part of $(a + b^{\frac{1}{2}})^{2m} + 1$ divisible by 2^{m+1} .

(1). The integral part of $(b^{\frac{1}{2}} + a)^{2m+1} = (b^{\frac{1}{2}} + a)^{2m+1} - (b^{\frac{1}{2}} - a)^{2m+1}$, provided $b^{\frac{1}{2}} - a < 1$ and > 0 , i.e. if b lie between a^2 and $(a + 1)^2$, which gives only $2a$ values of b .

Now by equation (Λ),

$$(b^{\frac{1}{2}} + a)^{2m+1} - (b^{\frac{1}{2}} - a)^{2m+1} = 2a \{ (b^{\frac{1}{2}} + a)^{2m} + (b^{\frac{1}{2}} - a)^{2m} \} \\ + (b - a^2) \{ (b^{\frac{1}{2}} + a)^{2m-2} + (b^{\frac{1}{2}} - a)^{2m-2} \} + \dots + (b - a^2)^m \}.$$

Now since b is only to have (a) values, we may make $b - a^2$ even, and then the problem is reduced to shewing that

$$(b^{\frac{1}{2}} + a)^{2n} + (b^{\frac{1}{2}} - a)^{2n}$$

is divisible by 2^n , which will *a fortiori* be true if

$$(b^{\frac{1}{2}} + a)^{2n} + (b^{\frac{1}{2}} - a)^{2n},$$

{the integral part of $(b^{\frac{1}{2}} + a)^{2n} + 1$ } be divisible by 2^{n+1} , and therefore (1) reduces itself to (3).

(2). The integral part of $(a + b^{\frac{1}{2}})^{2m+1} + 1 = (a + b^{\frac{1}{2}})^{2m+1} + (a - b^{\frac{1}{2}})^{2m+1}$, provided $a - b^{\frac{1}{2}} < 1$ and > 0 ; therefore b must lie between a^2 and $(a - 1)^2$, and it will appear by a precisely similar process to the above that (2) reduces itself to (3).

(3). The integral part of $(a + b^{\frac{1}{2}})^{2m} + 1 = (a + b^{\frac{1}{2}})^{2m} + (a - b^{\frac{1}{2}})^{2m}$, provided $a - b^{\frac{1}{2}} < 1$ and > -1 , *i.e.* if b lie between $(a - 1)^2$ and $(a + 1)^2$, giving only (4a) admissible values of b .

And if we further make $b - a^2$ or $a^2 - b$ even, the number of values of b will be reduced to (2a).

Then, these conditions being satisfied, (3) can be proved as in (a), only writing $b^{\frac{1}{2}}$ for $3^{\frac{1}{2}}$ and a for 1.

Hence the propositions enunciated are true.

1849.

1. A quantity of corn is to be divided amongst n persons, and is calculated to last a certain time if each of them receive a peck every week; during the distribution, it is found that one person dies every week and then the corn lasts twice as long as was expected: find the quantity of corn and the time that it lasts.

Let x = the number of pecks of corn,

y =weeks it is expected to last,

then $\frac{x}{y}$ = the whole number of persons = n ;

or $x = ny$ (1).

Also n = number of pecks distributed in the 1st week,

$n - 1$ = 2nd,

..... =

$n - r + 1$ = r^{th} ,

and it lasts $2y$ weeks; therefore

$n + (n - 1) + (n - 2) + \dots + (n - 2y + 1) = \text{whole quantity of corn,}$
 $= x,$

$$\therefore \frac{2(n-y)+1}{2} 2y = x,$$

$$\text{or } \{(2(n-y) + 1) y = x \dots \dots \dots (2);$$

$$\therefore \text{by (1) } 2(n-y) + 1 = n,$$

$$\therefore y = \frac{n+1}{2},$$

$$x = \frac{n(n+1)}{2},$$

$$\text{and } 2y = n + 1,$$

which determines the quantity of corn, and the time it lasts.

2. If ${}_m C_r$ be the number of combinations of m things taken r together, and p be less than m and n , shew that

$$\sum_{r=0}^{r=p} \{({}_m C_r) \cdot ({}_n C_{p-r})\} = {}_{(m+n)} C_p.$$

We have $(1+x)^m (1+x)^n = (1+x)^{m+n}$.

Now the coefficient of x^u in $(1+x)^p$ is ${}_p C_u$: also the coefficient of x^p in $(1+x)^{m+n}$ must be the sum of the products of the coefficients of x^r in $(1+x)^m$ multiplied into the coefficient of x^{p-r} in $(1+x)^n$ taken for all values of r from 0 to p . Hence

$$\sum_{r=0}^{r=p} \{({}_m C_r) \cdot ({}_n C_{p-r})\} = {}_{(m+n)} C_p,$$

3. If ${}_n C_r$ be the number of combinations of n things taken r together, prove that if a be an integer greater than 1, then will ${}_n C_{ra}$ be greater than $({}_n C_r)^a$.

The total number of combinations which can be made out of na things is ${}_{na}C_{ra}$. But if we divide the na things into a sets of n each, and restrict ourselves to those combinations containing ra things which can be made by taking r out of each set, we shall get, since ${}_nC_r$ combinations of r things may be made out of each set of n things, $({}_nC_r)^a$ combinations.

Hence, since ${}_{na}C_{ra}$ includes every possible mode of formations, and $({}_nC_r)^a$ only one particular one, it is clear that

$${}_{na}C_{ra} > ({}_nC_r)^a.$$

4. If p be greater than unity,

$$\frac{1}{p-1} = \frac{1}{p+1} + \frac{1.2}{(p+1)(p+2)} + \frac{1.2.3}{(p+1)(p+2)(p+3)} + \dots;$$

and if it be less,

$$\frac{1}{1-p} = \frac{1}{1+p} + \frac{1.2}{(1+p)(1+2p)}p + \frac{1.2.3}{(1+p)(1+2p)(1+3p)}p^2 + \dots$$

We have in general, when p is greater than unity,

$$\frac{1}{p-1} - \frac{1}{p+n} = \frac{n+1}{p+n} \cdot \frac{1}{p-1},$$

$$\therefore \frac{1}{p-1} = \frac{1}{p+n} + \frac{n+1}{p+n} \cdot \frac{1}{p-1};$$

therefore, putting successively $n = 1, 2, 3, \dots$,

$$\begin{aligned} \frac{1}{p-1} &= \frac{1}{p+1} + \frac{2}{p+1} \cdot \frac{1}{p-1}, \\ &= \frac{1}{p+1} + \frac{2}{p+1} \left(\frac{1}{p+2} + \frac{3}{p+2} \cdot \frac{1}{p-1} \right), \\ &= \frac{1}{p+1} + \frac{1.2}{(p+1)(p+2)} + \frac{1.2.3}{(p+1)(p+2)} \cdot \frac{1}{p-1}, \\ &= \frac{1}{p+1} + \frac{1.2}{(p+1)(p+2)} + \frac{1.2.3}{(p+1)(p+2)(p+3)} + \dots \end{aligned}$$

If p be less than unity,

$$\frac{1}{1-p} - \frac{1}{1+np} = \frac{(n+1)p}{1+np} \cdot \frac{1}{1-p};$$

therefore, putting $n = 1, 2, 3, \dots$ successively,

$$\begin{aligned} \frac{1}{1-p} &= \frac{1}{1+p} + \frac{2p}{1+p} \cdot \frac{1}{1-p}, \\ &= \frac{1}{1+p} + \frac{2p}{1+p} \left(\frac{1}{1+2p} + \frac{3p}{1+2p} \cdot \frac{1}{1-p} \right), \\ &= \frac{1}{1+p} + \frac{1.2}{(1+p)(1+2p)} p + \frac{1.2.3}{(1+p)(1+2p)(1+3p)} p^2 \\ &\quad + \dots \end{aligned}$$

5. A bag contains three bank-notes, and it is known that each of them is either a £5, £10, or £20 note; at three successive dips into the bag (replacing the note after each dip) a £5 note was drawn: what is the probable value of the contents of the bag?

There are six possible states of the bag, viz.

(1). 3 £5 notes in which case the value would be £15.

(2). 2 £5 and 1 £10..... £20.

(3). 2 £5 and 1 £20..... £30.

(4). 1 £5 and 2 £10..... £25.

(5). 1 £5 and 2 £20..... £45.

(6). 1 £5, 1 £10, and 1 £20..... £35,

and these are all *à priori* equally possible.

Now the chance of the observed event in case (1) is 1,

..... (2) or (3) $\left(\frac{2}{3}\right)^3$ or $\frac{8}{27}$,

..... (4) (5) or (6) $\left(\frac{1}{3}\right)^3$ or $\frac{1}{27}$,

therefore the probable value of the contents is

$$\begin{aligned} &\frac{1 \times £15 + \frac{8}{27} (£20 + £30) + \frac{1}{27} (£25 + £15 + £35)}{1 + 2 \times \frac{8}{27} + 3 \times \frac{1}{27}}, \\ &= £ \frac{15 + \frac{400}{27} + \frac{105}{27}}{\frac{46}{27}}, \\ &= £ \frac{910}{46}, \\ &= £19 \text{ 15s. } 7\frac{19}{23}d. \end{aligned}$$

1850.

1. Prove that the sum of the fractions which are intermediate in magnitude to any two numbers m and n , and have 3 for a denominator, is $n^2 - m^2$.

The fractions, together with the intermediate whole numbers, will be

$$\frac{3m+1}{3}, \frac{3m+2}{3}, \dots, \frac{3n-1}{3},$$

whose sum is
$$\left(\frac{3m+1}{3} + \frac{3n-1}{3} \right) \frac{(3n-1) - 3m}{2},$$

$$= \frac{(m+n) \{3(n-m) - 1\}}{2},$$

and the sum of the intermediate whole numbers is

$$(m+1) + (m+2) + \dots + (n-1),$$

$$= \{(m+1) + (n-1)\} \frac{n-1-m}{2},$$

$$= \frac{(m+n)(n-m-1)}{2};$$

therefore the sum of the fractions is

$$\frac{(m+n) \{3(n-m) - 1\}}{2} - \frac{(m+n)(n-m-1)}{2},$$

$$= n^2 - m^2.$$

2. There are a number of counters in a bag, of which one is marked 1, two marked 2,..... up to r marked r ; a person draws a counter at random, for which he is to receive as many shillings as the number marked on it: find the value of his expectation.

Since the person is as likely to draw one counter as another, the value of his expectation

$$= \frac{\text{total value of contents of bag}}{\text{number of counters in bag}},$$

$$= \frac{1^2 + 2^2 + \dots + r^2}{1 + 2 + \dots + r} \text{ shillings.}$$

$$\text{Now} \quad 1^2 + 2^2 + \dots + r^2 = \frac{r(r+1)(2r+1)}{6},$$

$$1 + 2 + \dots + r = \frac{r(r+1)}{2};$$

$$\begin{aligned} \text{hence his expectation} &= \frac{r(r+1)(2r+1)}{6} \div \frac{r(r+1)}{2}, \\ &= \frac{2r+1}{3} \text{ shillings.} \end{aligned}$$

3. If a, b, c be in harmonic progression, shew that

$$\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0.$$

$$\text{Let} \quad b = \frac{1}{\beta}, \quad a = \frac{1}{\beta + \alpha}, \quad c = \frac{1}{\beta - \alpha},$$

$$\text{then} \quad \frac{1}{a} + \frac{1}{c} = 2\beta,$$

$$\text{and} \quad a - b = \frac{1}{\beta + \alpha} - \frac{1}{\beta},$$

$$= -\frac{\alpha}{\beta(\beta + \alpha)},$$

$$c - b = \frac{1}{\beta - \alpha} - \frac{1}{\beta},$$

$$= \frac{\alpha}{\beta(\beta - \alpha)},$$

$$\therefore \frac{1}{a-b} + \frac{1}{c-b} = \frac{\beta(\beta - \alpha)}{\alpha} - \frac{\beta(\beta + \alpha)}{\alpha},$$

$$= -2\beta,$$

$$\therefore \frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0.$$

4. If there be z counters of which z_m are marked m ; z_n, n, \dots with or without other marks; $z_{m,n}, m, n$, with or without other marks; $z_{m,n,p,q}, \dots$ marked m, n, p, q, \dots ; the number unmarked is $z - \Sigma z_m + \Sigma z_{m,n} - \Sigma z_{m,n,p} + \dots$, Σ involving all combinations.

Let D_m denote the operation of selecting from the counters those marked with m , D_n those marked with n , &c. Then it is manifestly the same thing whether we first select from the heap those marked m , and then from these, those also marked n ; or whether we first select those marked n , then from these, those marked m ; or at once select those marked m, n . This may be symbolically expressed thus:

$$D_m D_n = D_n D_m = D_{m,n};$$

similarly, we have in general

$$D_m D_n D_p \dots = D_n D_m D_p \dots = \dots = D_{m,n,p} \dots; \dots\dots\dots(1).$$

Also $1 - D_m$ will denote selecting those unmarked with m , $1 - D_n$ those unmarked with n , &c. Hence the whole number unmarked

$$\begin{aligned} &= (1 - D_m)(1 - D_n) \dots z, \\ &= (1 - \Sigma D_m + \Sigma D_{m,n} - \Sigma D_{m,n,p} + \dots) z; \end{aligned}$$

since the symbols D_m, D_n, \dots have been shewn (equation (1)) to be commutative

$$= z - \Sigma z_m + \Sigma z_{m,n} - \Sigma z_{m,n,p} + \dots,$$

the required number.

5. Prove that

$$x^8 + y^8 + (x+y)^8 = 2(x^2 + xy + y^2)^4 + 8x^2 y^2 (x+y)^2 (x^2 + xy + y^2);$$

and if $x^2 + xy + y^2 = a$, $xy(x+y) = b$, and n be any positive integer, shew that

$$\begin{aligned} x^{2n} + y^{2n} + (x+y)^{2n} &= 2a^n + n(n-2)a^{n-3}b^2 + \frac{n(n-3)(n-4)(n-5)}{3.4} a^{n-6}b^4 \\ &+ \dots + \frac{n(n-r-1)(n-r-2)\dots(n-3r+1)}{3.4\dots 2r} a^{n-3r}b^{2r} + \dots \end{aligned}$$

(α). Let z be a quantity, such that

$$x + y + z = 0;$$

$$\text{then } x^{2n} + y^{2n} + (x+y)^{2n} = x^{2n} + y^{2n} + z^{2n},$$

$$\begin{aligned}\text{and } x^2 + xy + y^2 &= (x+y)^2 - xy, \\ &= -z(x+y) - xy, \\ &= -(yz + zx + xy),\end{aligned}$$

$$xy(x+y) = -xyz;$$

therefore, taking the notation of the latter part of the question,

$$yz + zx + xy = -a,$$

$$xyz = -b,$$

therefore x, y, z , are the roots of the equation

$$\xi^3 - a\xi + b = 0;$$

$$\therefore (\xi - x)(\xi - y)(\xi - z) = \xi^3 - a\xi + b, \text{ identically,}$$

$$\begin{aligned}\therefore \left(1 - \frac{x}{\xi}\right) \left(1 - \frac{y}{\xi}\right) \left(1 - \frac{z}{\xi}\right) &= 1 - \frac{a}{\xi^2} + \frac{b}{\xi^3}, \\ &= 1 - \frac{1}{\xi^2} \left(a - \frac{b}{\xi}\right),\end{aligned}$$

$$\therefore \log \left(1 - \frac{x}{\xi}\right) + \log \left(1 - \frac{y}{\xi}\right) + \log \left(1 - \frac{z}{\xi}\right) = \log \left\{1 - \frac{1}{\xi^2} \left(a - \frac{b}{\xi}\right)\right\};$$

$$\begin{aligned}\therefore \frac{x+y+z}{\xi} + \frac{1}{2} \frac{x^2+y^2+z^2}{\xi^2} + \dots + \frac{1}{2n} \frac{x^{2n}+y^{2n}+z^{2n}}{\xi^{2n}} + \dots \\ = \frac{1}{\xi^2} \left(a - \frac{b}{\xi}\right) + \frac{1}{2\xi^n} \left(a - \frac{b}{\xi}\right)^2 + \dots + \frac{1}{n\xi^{2n}} \left(a - \frac{b}{\xi}\right)^n + \dots;\end{aligned}$$

therefore equating coefficients of $\frac{1}{\xi^{2n}}$,

$$\begin{aligned}\frac{1}{2n} (x^{2n} + y^{2n} + z^{2n}) &= \frac{1}{n} a^n + \frac{1}{n-1} \frac{(n-1)(n-2)}{1.2} a^{n-3} b^2 \\ &\quad + \frac{1}{n-2} \frac{(n-2)(n-3)(n-4)(n-5)}{1.2.3.4} a^{n-6} b^4 + \dots \\ &\quad + \frac{1}{n-r} \frac{(n-r)(n-r-1)\dots(n-3r+1)}{1.2\dots 2r} a^{n-3r} b^{2r} + \dots \\ &= \frac{a^n}{n} + \frac{n-2}{2} a^{n-3} b^2 + \frac{(n-3)(n-4)(n-5)}{2.3.4} a^{n-6} b^4 + \dots \\ &\quad + \frac{(n-r-1)\dots(n-3r+1)}{2.3\dots 2n} a^{n-3r} b^{2r} + \dots;\end{aligned}$$

$$\begin{aligned} \therefore x^{2n} + y^{2n} + (x+y)^{2n} &= 2a^n + n(n-2)a^{n-3}b^2 \\ &\quad + \frac{n(n-3)(n-4)(n-5)}{3.4} a^{n-6}b^4 + \dots \\ &\quad + \frac{n(n-r-1)\dots(n-3r+1)}{3.4\dots 2r} a^{n-3r}b^{2r} + \dots \end{aligned}$$

Hence putting $n = 4$, we get

$$x^8 + y^8 + (x+y)^8 = 2(x^2 + xy + y^2)^2 + 8(x^2 + xy + y^2)x^2y^2(x+y)^2.$$

6. (α). If a be less than b , prove that $\left(\frac{a}{b}\right)^{a+b}$ is increased by adding the same quantity to a and b .

(β). And if n be greater than 1, shew that $\left(\frac{a}{b}\right)^n > \left(\frac{na-b}{nb-b}\right)^{n-1}$; by means of this formula prove that

$$(a_1 + a_2 + \dots + a_n)^n > n^n a_1 a_2 \dots a_n.$$

(α). Since a is less than b , we may put $a = b - c$, where c is a positive quantity less than b ; then a and b will be increased by the same quantity if we increase b , c remaining constant;

$$\therefore \left(\frac{a}{b}\right)^{a+b} = \left(1 - \frac{c}{b}\right)^{2b-c} = x \text{ suppose;}$$

$$\therefore \log x = (2b - c) \log \left(1 - \frac{c}{b}\right),$$

$$= -(2b - c) \left(\frac{c}{b} + \frac{1}{2} \frac{c^2}{b^2} + \frac{1}{3} \frac{c^3}{b^3} + \dots \right),$$

$$= -c \left(2 - \frac{c}{b} \right) \left(1 + \frac{1}{2} \frac{c}{b} + \frac{1}{3} \frac{c^2}{b^2} + \dots \right),$$

$$= -c \left\{ 2 + \left(\frac{2}{2} - \frac{1}{3} \right) \frac{c}{b} + \left(\frac{2}{3} - \frac{1}{4} \right) \frac{c^2}{b^2} + \dots + \left(\frac{2}{n} - \frac{1}{n+1} \right) \frac{c^{n-1}}{b^{n-1}} + \dots \right\},$$

which is manifestly negative, since the coefficients being of the form $\left(\frac{2}{n} - \frac{1}{n+1}\right)$ are positive; and the absolute magnitude of the series is diminished by increasing b if c remain unaltered; hence $\log x$, and therefore x or $\left(\frac{a}{b}\right)^{a+b}$, is increased by adding the same quantity to a and b .

(β).^{*} First, let $\frac{a}{b}$ be greater than unity $= 1 + \frac{x}{n}$ suppose; we have then to shew that

$$\left(1 + \frac{x}{n}\right)^n > \left\{ \frac{n \left(1 + \frac{x}{n}\right) - 1}{n-1} \right\}^{n-1} \quad \text{or} \quad \left(1 + \frac{x}{n-1}\right)^{n-1} :$$

these quantities when expanded by the Binomial Theorem become

$$1 + x + \frac{1 - \frac{1}{n}}{1.2} x^2 + \frac{\left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right)}{1.2.3} x^3 + \dots,$$

$$\text{and } 1 + x + \frac{1 - \frac{1}{n-1}}{1.2} x^2 + \frac{\left(1 - \frac{1}{n-1}\right) \left(1 - \frac{2}{n-1}\right)}{1.2.3} x^3 + \dots;$$

all the terms of both series are positive, and each term of the first series greater than the corresponding term of the second, or

$$\left(1 + \frac{x}{n}\right)^n > \left(1 + \frac{x}{n-1}\right)^n \quad \text{and} \quad \left(\frac{a}{b}\right)^n > \left(\frac{na-b}{nb-b}\right)^{n-1},$$

when a is greater than b .

Next, suppose $\frac{a}{b}$ less than unity $= 1 - \frac{x}{n}$ suppose, we have to shew that

$$\left(1 - \frac{x}{n}\right)^n > \left\{ \frac{2 \left(1 - \frac{x}{n}\right) - 1}{n-1} \right\}^{n-1} \quad \text{or} \quad \left(1 - \frac{x}{n-1}\right)^{n-1} ;$$

$$\text{or that } \left(1 - \frac{x}{n-1}\right)^{-\overline{n-1}} > (1-x)^{-n} :$$

these quantities when expanded by the binomial theorem become

$$1 + x + \frac{1 + \frac{1}{n-1}}{1.2} x^2 + \frac{\left(1 + \frac{1}{n-1}\right) \left(1 + \frac{2}{n-1}\right)}{1.2.3} x^3 + \dots,$$

$$\text{and } 1 + x + \frac{1 + \frac{1}{n}}{1.2} x^2 + \frac{\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)}{1.2.3} x^3 + \dots;$$

^{*} This part of the solution is given by Mr. Thacker in a recent number of the *Cambridge and Dublin Mathematical Journal*, No. xxv. p. 81.

all the terms of both series are positive, and each term of the first series greater than the corresponding term of the second, and therefore the proposition is true in this case also.

Now let the n quantities a_1, a_2, \dots, a_n , be in ascending order of magnitude; then

$$\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^n = a_1^n \left(1 + \frac{a_1 + a_2 + \dots + a_n - na_1}{na_1}\right)^n,$$

$$> a_1^n \left\{1 + \frac{a_1 + a_2 + \dots + a_n - na_1}{(n-1)a_1}\right\}^{n-1},$$

by the above,

$$> a_1 \left(\frac{a_2 + a_3 + \dots + a_n}{n-1}\right)^{n-1};$$

$$\text{similarly } \left(\frac{a_2 + a_3 + \dots + a_n}{n-1}\right)^{n-1} > a_2 \left(\frac{a_3 + a_4 + \dots + a_n}{n-2}\right)^{n-2},$$

$$\dots\dots\dots > \dots\dots\dots,$$

$$\left(\frac{a_{n-1} + a_n}{2}\right)^2 > a_{n-1}a_n;$$

hence by multiplication

$$\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^n > a_1 a_2 \dots a_n,$$

$$\text{or } (a_1 + a_2 + \dots + a_n)^n > n^n (a_1 a_2 \dots a_n).$$

7. If $\frac{p}{q}$ be the r^{th} fraction converging to $\frac{m}{n}$, and n' be the r^{th} remainder in the process for finding the successive quotients, prove that

$$\frac{m}{n} \sim \frac{p}{q} = \frac{p'}{pq}.$$

Let $\frac{p_{r-2}}{q_{r-2}}, \frac{p_{r-1}}{q_{r-1}}$ be the $(r-2)^{\text{th}}$ and $(r-1)^{\text{th}}$ converging fractions respectively; m' the r^{th} quotient, n'' the $(r-1)^{\text{th}}$ remainder; then

$$p = m'p_{r-1} + p_{r-2},$$

$$q = m'q_{r-1} + q_{r-2}$$

And the fraction $\frac{m}{n}$ may be derived from $\frac{p}{q}$ by writing $m' + \frac{n'}{n''}$ for m' in the above expressions for p and q ;

$$\therefore \frac{m}{n} = \frac{\left(m' + \frac{n'}{n''}\right) p_{r-1} + p_{r-2}}{\left(m' + \frac{n'}{n''}\right) q_{r-1} + q_{r-2}}.$$

Now $\frac{m}{n}$ is in its lowest terms, and n' is prime to n'' ,

$$\therefore m = (m'n'' + n') p_{r-1} + n'' p_{r-2},$$

$$n = (m'n'' + n') q_{r-1} + n'' q_{r-2},$$

$$\begin{aligned} \text{and } \frac{m}{n} \sim \frac{p}{q} &= \frac{mq \sim np}{nq}, \\ &= \frac{n' p_{r-1} q_{r-2} \sim n' p_{r-2} q_{r-1}}{nq}, \\ &= \frac{n'}{nq}. \end{aligned}$$

8. Find the probability of drawing a black and a white ball the same number of times from a bag which contains an equal number of each; the balls being drawn one by one and replaced after each drawing, and the number of drawings being the same as the number of balls in the bag, but this number is unknown, any number from 2 to $2n$ being equally probable.

Suppose there are $2x$ balls in the bag; the number of drawings will then be $2x$, and the number of possible ways in which the balls may come out $= 2^{2x}$.

Of these the number of favourable cases equals the number of permutations of $2x$ things taken all together, of which x are of one kind and x of another,

$$\begin{aligned} &= \frac{1.2 \dots 2x}{(1.2 \dots x)^2}, \\ &= \frac{(x+1)(x+2) \dots 2x}{1.2 \dots x}, \end{aligned}$$

therefore chance of proposed event on this supposition

$$= \frac{1}{2^{2x}} \frac{(x+1)(x+2) \dots 2x}{1.2 \dots x}.$$

Now the chance of there being $2x$ balls in the bag $= \frac{1}{n}$ whatever number > 0 , $\nless n$, x may be. Hence the chance of the proposed event

$$= \frac{1}{n} \left\{ \frac{1}{2^2} \frac{2}{1} + \frac{1}{2^4} \frac{3.4}{1.2} + \dots + \frac{1}{2^{2n}} \frac{(n+1)(n+2)\dots 2n}{1.2\dots n} \right\}.$$

1851.

1. If $\frac{a}{b}$, $\frac{a'}{b'}$ be fractions in their least terms, the denominators of which do not exceed a given number n , the former fraction being given, and the latter determined from it by taking for a' and b' the greatest values of x and y (y not greater than n) which satisfy the equation $bx - ay = 1$, then of all the fraction in their least terms, the denominators of which do not exceed n , the fraction $\frac{a'}{b'}$ exceeds $\frac{a}{b}$ by the smallest quantity.

We have
$$\frac{a'}{b'} - \frac{a}{b} = \frac{ba' - ab'}{bb'} = \frac{1}{bb'},$$

since a' , b' are values of x and y in the equation $bx - ay = 1$.

Let $\frac{\alpha}{\beta}$ be any other fraction in its least terms whose denominator does not exceed n , then

$$\frac{\alpha}{\beta} - \frac{a}{b} = \frac{b\alpha - a\beta}{b\beta} = \frac{m}{b\beta} \text{ say,}$$

m being some integer greater than 1; then α and β are values of x and y in the equation

$$bx - ay = m.$$

Now this equation is satisfied by

$$x = ma' \pm qa,$$

$$y = mb' \pm qb,$$

where q is any integer.

Again, the successive values of x and y which satisfy the equation

$$bx - ay = 1,$$

differ by a and b respectively; hence, as b' is the greatest value of y , less than n , in this equation, $b' + b > n$, therefore *à fortiori* $mb' + b > n$. Hence β cannot be of the form $mb' + qb$, since it is less than n .

Neither can it be of the form mb' ; for then α would $= ma'$, and $\frac{\alpha}{\beta}$ would not be in its least terms.

Hence β must be of the form $mb' - qb$:

$$\therefore \beta b = (mb' - qb)b < mbb';$$

$$\begin{aligned}\therefore \frac{\alpha}{\beta} - \frac{a}{b} &> \frac{m}{mbb'} > \frac{1}{bb'}, \\ &> \frac{a'}{b'} - \frac{a}{b};\end{aligned}$$

or of all the fractions in their least terms, whose denominators do not exceed n , $\frac{a'}{b'}$ exceeds $\frac{a}{b}$ by the smallest quantity.

2. The sum of the series $\frac{1}{1^{1+\delta}} + \frac{1}{2^{1+\delta}} + \frac{1}{3^{1+\delta}} + \dots$ to infinity, where δ is positive, is greater than $(2^\delta - \frac{1}{2}) \div (2^\delta - 1)$, but less than $2^\delta \div (2^\delta - 1)$.

Let S be the sum of the given series.

Then

$$\begin{aligned}S &= \frac{1}{1^{1+\delta}} + \frac{1}{2^{1+\delta}} + \left(\frac{1}{3^{1+\delta}} + \frac{1}{4^{1+\delta}} \right) + \left(\frac{1}{5^{1+\delta}} + \frac{1}{6^{1+\delta}} + \frac{1}{7^{1+\delta}} + \frac{1}{8^{1+\delta}} \right) + \dots \\ &> \frac{1}{1^{1+\delta}} + \frac{1}{2^{1+\delta}} + \left(\frac{1}{4^{1+\delta}} + \frac{1}{4^{1+\delta}} \right) + \left(\frac{1}{8^{1+\delta}} + \frac{1}{8^{1+\delta}} + \frac{1}{8^{1+\delta}} + \frac{1}{8^{1+\delta}} \right) + \dots \\ &> \frac{1}{1^{1+\delta}} + \frac{1}{2} \left(\frac{2}{2^{1+\delta}} + \frac{4}{4^{1+\delta}} + \frac{8}{8^{1+\delta}} + \dots \right) \\ &> \frac{1}{1^\delta} + \frac{1}{2} \left(\frac{1}{2^\delta} + \frac{1}{4^\delta} + \frac{1}{8^\delta} + \dots \right) \\ &> 1 + \frac{1}{2} \frac{1}{2^\delta - 1} \\ &> \frac{2^\delta - \frac{1}{2}}{2^\delta - 1}.\end{aligned}$$

Again,

$$\begin{aligned}
 S &= \frac{1}{1^{1+\delta}} + \left(\frac{1}{2^{1+\delta}} + \frac{1}{3^{1+\delta}} \right) + \left(\frac{1}{4^{1+\delta}} + \frac{1}{5^{1+\delta}} + \frac{1}{6^{1+\delta}} + \frac{1}{7^{1+\delta}} \right) + \dots \\
 &< \frac{1}{1^{1+\delta}} + \left(\frac{1}{2^{1+\delta}} + \frac{1}{2^{1+\delta}} \right) + \left(\frac{1}{4^{1+\delta}} + \frac{1}{4^{1+\delta}} + \frac{1}{4^{1+\delta}} + \frac{1}{4^{1+\delta}} \right) \\
 &< \frac{1}{1^\delta} + \frac{1}{2^\delta} + \frac{1}{4^\delta} + \dots \\
 &< \frac{1}{1 - \frac{1}{2^\delta}} \\
 &< \frac{2^\delta}{2^\delta - 1}.
 \end{aligned}$$

Hence S lies between $\frac{2^\delta - \frac{1}{2}}{2^\delta - 1}$ and $\frac{2^\delta}{2^\delta - 1}$.

3. Solve the equation

$$x + \frac{1}{1-x} - \frac{1-x}{x} - a - \frac{1}{1-a} + \frac{1-a}{a} = 0;$$

and thence infer the resolution of the first side of the equation into factors.

We see at once that the given equation is satisfied by $x = a$.

Again, for a write $\frac{1}{1-a'}$, then

$$\begin{aligned}
 \frac{1}{1-a} &= \frac{1}{1 - \frac{1}{1-a'}} = -\frac{1-a'}{a'}, \\
 \frac{1-a}{a} &= \frac{1 - \frac{1}{1-a'}}{\frac{1}{1-a'}} = -a'.
 \end{aligned}$$

Hence the given equation becomes

$$x + \frac{1}{1-x} - \frac{1-x}{x} - a' - \frac{1}{1-a'} + \frac{1-a'}{a'} = 0,$$

of which a' or $\frac{a-1}{a}$ is a root.

Similarly, $\frac{a' - 1}{a'}$ is a root of the given equation.

$$\text{But } \frac{a' - 1}{a'} = \frac{\frac{a - 1}{a} - 1}{\frac{a - 1}{a}} = \frac{1}{1 - a};$$

therefore the roots of the given equation are a , $\frac{1}{1 - a}$, $-\frac{1 - a}{a}$;

and the first side may therefore be put into the form

$$-\frac{(x - a) \left(x - \frac{1}{1 - a}\right) \left(x + \frac{1 - a}{a}\right)}{x(1 - x)}.$$

4. From the equation

$$ab(c + d - e - f) + cd(e + f - a - b) + ef(a + b - c - d) = 0,$$

determine, in terms of b, c, d, e, f , the ratios $a - c : a - d$, and $a - e : a - f$; and shew that the relation between the six letters may be expressed in the form $P = Q$, where P and Q are each of them the product of three differences of pairs of letters, or in the form $R = S$, where R and S are each of them the product of four differences of pairs of letters.

(α) From the given equation we get

$$\begin{aligned} a &= \frac{cd(b - e - f) + ef(c + d - b)}{b(c + d - e - f) - cd + ef}; \\ \therefore a - c &= \frac{c(b - d)(e + f) - bc^2 + ef(d - b) + c^2d}{b(c + d - e - f) - cd + ef} \\ &= (b - d) \frac{c(e + f) - c^2 - ef}{b(c + d - e - f) - cd + ef} \\ &= - (b - d) \frac{(c - e)(c - f)}{b(c + d - e - f) - cd + ef} \dots\dots (1). \end{aligned}$$

Similarly, $a - d = - (b - c) \frac{(d - e)(d - f)}{b(c + d - e - f) - cd + ef}$;

$$\therefore \frac{a - c}{a - d} = \frac{b - d}{b - c} \cdot \frac{c - e}{d - e} \cdot \frac{c - f}{d - f} \dots\dots\dots (2).$$

In the same manner it may be shewn that

$$\frac{a-e}{a-f} = \frac{b-f}{b-e} \cdot \frac{c-e}{f-e} \cdot \frac{e-d}{f-d} \dots\dots\dots (3).$$

(β) From (1) we see, by interchanging c with e , d with f , which does not alter the original equation, that

$$\begin{aligned} a-e &= -(b-f) \frac{(e-c)(e-d)}{b(e+f-c-d) + ef - cd}; \\ \therefore \frac{a-c}{a-e} &= -\frac{b-d}{b-f} \frac{(c-e)(c-f)}{(e-c)(e-d)}; \\ \therefore (a-c)(b-f)(e-d) &= (b-d)(c-f)(a-e), \end{aligned}$$

which expresses the relation between the six letters in the form $P = Q$.

(γ) Again, by (2) we get

$$(a-c)(b-e)(d-f) = (a-d)(b-d)(c-e)(c-f),$$

which is in the form $R = S$.

5. If $a^2 + 1$ be exactly divisible by p , and $\frac{a}{p}$ be converted into a continued fraction, until two consecutive reduced fractions, $\frac{m}{n}$, $\frac{m'}{n'}$, are found, such that $p^{\frac{1}{2}} > n < n'$, then

$$p = (na - mp)^2 + n^2.$$

It is manifest that

$$\begin{aligned} (na - mp)^2 &= n^2 p^2 \left(\frac{a}{p} - \frac{m}{n} \right)^2 \\ &< n^2 p^2 \left(\frac{m'}{n'} - \frac{m}{n} \right)^2, \end{aligned}$$

by the property of Continued Fractions.

But
$$\frac{m'}{n'} - \frac{m}{n} = \pm \frac{1}{nn'};$$

$$\therefore (na - mp)^2 < \frac{p^2}{n'^2};$$

and

$$n'^2 > p;$$

therefore, *à fortiori*, $(na - mp)^2 < p$,

and $n^2 < p$;

$$\therefore (na - mp)^2 + n^2 < 2p.$$

But $(na - mp)^2 + n^2 = n^2(a^2 + 1) - 2mna.p + m^2p^2$,

which must be divisible by p , since $a^2 + 1$ is so :

and $(na - mp)^2 + n^2$ has been shewn to be $< 2p$, and it must be positive; therefore we must have

$$(na - mp)^2 + n^2 = p.$$

6. If $\{\alpha + \beta + \gamma + \dots\}^p$ denote the expansion of $(\alpha + \beta + \gamma + \dots)^p$, retaining those terms $N\alpha^a\beta^b\gamma^c\delta^d\dots$ only in which

$$b + c + d + \dots \geq p - 1, \quad c + d + \dots \geq p - 2, \quad \&c. \quad \&c.,$$

then

$$\begin{aligned} x^n &= (x + \alpha)^n - n\{\alpha\}^1(x + \alpha + \beta)^{n-1} + \frac{n(n-1)}{1.2} \{\alpha + \beta\}^2(x + \alpha + \beta + \gamma)^{n-2} \\ &\quad - \frac{n(n-1)(n-2)}{1.2.3} \{\alpha + \beta + \gamma\}^3(x + \alpha + \beta + \gamma + \delta)^{n-3} + \dots \end{aligned}$$

*This theorem may be put into a rather more convenient form by writing $x - \alpha$ for x ; we have then to shew that

$$\begin{aligned} (x - \alpha)^n &= x^n - n\{\alpha\}^1(x + \beta)^{n-1} + \frac{n(n-1)}{1.2} \{\alpha + \beta\}^2(x + \beta + \gamma)^{n-2} \\ &\quad - \frac{n(n-1)(n-2)}{1.2.3} \{\alpha + \beta + \gamma\}^3(x + \beta + \gamma + \delta)^{n-3} + \dots; \end{aligned}$$

or, writing α_1 for α , α_2 for β , &c.,

$$\begin{aligned} (x - \alpha_1)^n &= x^n - n\{\alpha_1\}^1(x + \alpha_2)^{n-1} + \frac{n(n-1)}{1.2} \{\alpha_1 + \alpha_2\}^2(x + \alpha_2 + \alpha_3)^{n-2} \\ &\quad - \frac{n(n-1)(n-2)}{1.2.3} \{\alpha_1 + \alpha_2 + \alpha_3\}^3(x + \alpha_2 + \alpha_3 + \alpha_4)^{n-3} + \dots \quad (1). \end{aligned}$$

The proof of this depends on the expansion of the quantity

$$\{\alpha_1 + \alpha_2 + \dots + \alpha_p\}^p.$$

* For the solution of this problem we are indebted to Mr. Cayley.

Expanding by the Binomial Theorem,

$$\begin{aligned}
 (\alpha_1 + \alpha_2 + \dots + \alpha_p)^p &= \alpha_1^p + \frac{p}{1} \alpha_1^{p-1} (\alpha_2 + \alpha_3 + \dots + \alpha_p)^1 \\
 &+ \frac{p}{1} \frac{p-1}{2} \alpha_1^{p-2} (\alpha_2 + \alpha_3 + \dots + \alpha_p)^2 + \dots \\
 &+ \frac{p}{1} \frac{p-1}{2} \dots \frac{p-r+1}{r} \alpha_1^{p-r} (\alpha_2 + \alpha_3 + \dots + \alpha_p)^r + \dots
 \end{aligned}$$

To pass to $\{\alpha_1 + \alpha_2 + \dots + \alpha_p\}^p$. The sum of the indices of $\alpha_2, \alpha_3, \dots, \alpha_p$, are not to exceed $p-1$..., and generally the sum of the indices of $\alpha_r, \alpha_{r+1}, \dots, \alpha_p$, are not to exceed $p-r+1$. Hence in $(\alpha_2 + \alpha_3 + \dots + \alpha_p)^r$, the required conditions will be satisfied, if only the sums of the indices of $\alpha_{p-r+2}, \alpha_{p-r+3}, \dots, \alpha_p$ do not exceed $r-1$, and the sum of the indices of $\alpha_{p-r+3}, \alpha_{p-r+4}, \dots, \alpha_p$ do not exceed $r-2$, and so on. And this will be the case if, *considering* $\alpha_2 + \alpha_3 + \dots + \alpha_{p-r+1}$ *as one quantity*, we replace $(\alpha_2 + \alpha_3 + \dots + \alpha_p)^r$ by

$$\{(\alpha_2 + \alpha_3 + \dots + \alpha_{p-r+1}) + \alpha_{p-r+2} + \dots + \alpha_p\}^r.$$

$$\begin{aligned}
 \text{Hence } \{\alpha_1 + \alpha_2 + \dots + \alpha_p\}^p &= \alpha_1^p + \frac{p}{1} \alpha_1^{p-1} \{(\alpha_2 + \alpha_3 + \dots + \alpha_{p-r+1} \alpha_p)\}^1 \\
 &+ \frac{p}{1} \frac{p-1}{2} \alpha_1^{p-2} \{(\alpha_2 + \alpha_3 + \dots + \alpha_{p-r}) + \alpha_{p-1}\}^2 + \dots \\
 &+ \frac{p}{1} \frac{p-1}{2} \dots \frac{p-r+1}{r} \alpha_1^{p-r} \{(\alpha_2 + \alpha_3 + \dots + \alpha_{p-r+1}) + \alpha_{p-r+2} + \dots + \alpha_p\}^r + \dots \\
 &+ p \alpha_1 \{\alpha_2 + \alpha_3 + \dots + \alpha_p\}^{p-1} \dots \dots \dots (2),
 \end{aligned}$$

the last term $(\alpha_2 + \alpha_3 + \dots + \alpha_p)^p$ being of course rejected altogether, since the sum of the indices exceeds $p-1$.

Now the coefficient of $\frac{n}{1} \frac{n-1}{2} \dots \frac{n-r+1}{r} (-\alpha_1)^r$, on the left-hand side of equation (1), is x^{n-r} .

On the right-hand side, it is, expanding each of the quantities in the brackets $\{\}$ by (2),

$$\begin{aligned}
 (x + \alpha_2 + \dots + \alpha_{r+1})^{n-r} &- \frac{n-r}{1} \{(\alpha_2 + \dots + \alpha_{r+1})\}^1 (x + \alpha_2 + \dots + \alpha_{r+2})^{n-r-1} \\
 &+ \frac{n-r}{1} \frac{n-r-1}{2} \{(\alpha_2 + \dots + \alpha_{r+1}) + \alpha_{r+2}\}^2 (x + \alpha_2 + \dots + \alpha_{r+3})^{n-r-2} - \dots
 \end{aligned}$$

or, if we write y for $x + \alpha_2 + \dots + \alpha_{r+1}$, ω for $\alpha_2 + \dots + \alpha_{r+1}$, the coefficient on one side is $(y - \omega)^{n-r}$, on the other

$$y^{n-r} - \frac{n-r}{1} \{\omega\}^1 (y + \alpha_{r+2})^{n-r-1} \\ + \frac{n-r}{1} \frac{n-r-1}{2} \{\omega + \alpha_{r+2}\}^2 (y + \alpha_{r+2} + \alpha_{r+3})^{n-r-2}, \\ + \dots$$

Therefore we have to shew that

$$(y - \omega)^{n-r} = y^{n-r} - \frac{n-r}{1} \{\omega\}^1 (y + \alpha_{r+2})^{n-r-1} \\ + \frac{n-r}{1} \frac{n-r-1}{2} \{\omega + \alpha_{r+2}\}^2 (y + \alpha_{r+2} + \alpha_{r+3})^{n-r-2} - \dots$$

the theorem itself, writing $n - r$ for n .

The coefficients of α_1^0 , on each side of the equation (1), are obviously equal. If then the theorem hold for the indices $1, 2 \dots (n-1)$, it is proved to hold for the index n . But it obviously holds for the index 1; therefore it holds universally.

$$7. \text{ If } \frac{1}{mp + x} = \alpha + \frac{\beta}{p + u}, \text{ and } \frac{1}{mp' + x} = \alpha' + \frac{\beta}{p' + u'},$$

then supposing u and u' to vanish when x vanishes,

$$\frac{1}{u'} - \frac{1}{u} = m(\alpha - \alpha').$$

Since u and u' vanish when x vanishes, we have

$$\frac{1}{mp} = \alpha + \frac{\beta}{p}, \quad \frac{1}{mp'} = \alpha' + \frac{\beta}{p'},$$

$$\therefore \alpha p = \alpha' p' = \frac{1}{m} - \beta.$$

Hence, by the first of the given equations,

$$\frac{p + u}{mp + x} = \alpha u + \frac{1}{m};$$

$$\therefore p + u = \alpha u (mp + x) + p + \frac{x}{m} \\ = u(1 - m\beta + \alpha x) + p \frac{x}{m};$$

$$\therefore u(m\beta - \alpha x) = \frac{x}{m},$$

$$\therefore \frac{m^2\beta}{x} - m\alpha = \frac{1}{u}.$$

Similarly, by the second of the given equations,

$$\frac{m^2\beta}{x} - m\alpha' = \frac{1}{u'};$$

$$\therefore \frac{1}{u'} - \frac{1}{u} = m(\alpha - \alpha'),$$

the required result.

PLANE TRIGONOMETRY.

1848.

A cannon-ball is moving in a direction making an acute angle θ with a line drawn from the ball to an observer; if V be the velocity of sound, and nV that of the ball, prove that the whizzing of the ball at the different points of its course will be heard in the order in which it is produced, or in the reverse order, according as $n < > \sec \theta$.

The whizzing will be heard in the order in which it is produced, or in the reverse order, according as the sound or the ball moves more quickly towards the observer. Now the velocity with which the ball moves towards the observer $= nV \cos \theta$; hence the whizzing will be heard in the natural or reversed order, according as

$$V > < nV \cos \theta,$$

or

$$n < > \sec \theta,$$

the required condition.

$$2. \text{ If } \frac{\sin(\alpha - \beta)}{\sin \beta} = \frac{\sin(\alpha + \theta)}{\sin \theta},$$

shew that $\cot \beta - \cot \theta = \cot(\alpha + \theta) + \cot(\alpha - \beta)$.

$$\text{Since } \frac{\sin(\alpha - \beta)}{\sin \beta} = \frac{\sin(\alpha + \theta)}{\sin \theta},$$

$$\therefore \frac{1}{\sin \beta \sin(\alpha + \theta)} = \frac{1}{\sin \theta \sin(\alpha - \beta)};$$

$$\therefore \frac{\sin(\alpha + \theta - \beta)}{\sin \beta \sin(\alpha + \theta)} = \frac{\sin(\alpha - \beta + \theta)}{\sin \theta \sin(\alpha - \beta)};$$

$$\therefore \cot \beta - \cot(\alpha + \theta) = \cot \theta + \cot(\alpha - \beta),$$

or

$$\cot \beta - \cot \theta = \cot(\alpha + \theta) + \cot(\alpha - \beta),$$

the required result.

1849.

1. If $\cos \alpha = \cos \beta \cos \phi = \cos \beta' \cos \phi'$,

and $\sin \alpha = 2 \sin \frac{1}{2} \phi \cdot \sin \frac{1}{2} \phi'$,

shew that $\tan \frac{1}{2} \alpha = \tan \frac{1}{2} \beta \tan \frac{1}{2} \beta'$.

Since $\sin \alpha = 2 \sin \frac{1}{2} \phi \sin \frac{1}{2} \phi'$,

$$\therefore \sin^2 \alpha = (2 \sin^2 \frac{1}{2} \phi) (2 \sin^2 \frac{1}{2} \phi')$$

$$= (1 - \cos \phi) (1 - \cos \phi');$$

$$\therefore 1 - \cos^2 \alpha = \left(1 - \frac{\cos \alpha}{\cos \beta}\right) \left(1 - \frac{\cos \alpha}{\cos \beta'}\right),$$

$$\therefore \cos \alpha = \sec \beta + \sec \beta' - \cos \alpha \sec \beta \sec \beta',$$

and $\cos \alpha = \frac{\sec \beta + \sec \beta'}{1 + \sec \beta \sec \beta'} = \frac{\cos \beta + \cos \beta'}{1 + \cos \beta \cos \beta'}$;

$$\therefore \tan^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{1 + \cos \alpha} = \frac{1 - \cos \beta - \cos \beta' + \cos \beta \cos \beta'}{1 + \cos \beta + \cos \beta' + \cos \beta \cos \beta'}$$

$$= \frac{(1 - \cos \beta) (1 - \cos \beta')}{(1 + \cos \beta) (1 + \cos \beta')} = \tan^2 \frac{\beta}{2} \tan^2 \frac{\beta'}{2},$$

and $\tan \frac{\alpha}{2} = \tan \frac{\beta}{2} \tan \frac{\beta'}{2}$.

2. Find $\frac{a}{b}$ from the equations

$$(a+b) \sin \theta + (a-b) \cos \theta = (a^2 + b^2)^{\frac{1}{2}},$$

$$a \sin^3 \theta + b \cos^3 \theta = (3ab)^{\frac{1}{2}}.$$

Squaring the first of the given equations, we get

$$(a^2 + b^2)(\sin^2 \theta + \cos^2 \theta) - 2ab(\cos^2 \theta - \sin^2 \theta) + 2(a^2 - b^2) \sin \theta \cos \theta = a^2 + b^2;$$

$$\therefore (a^2 - b^2) \sin 2\theta - 2ab \cos 2\theta = 0.$$

Let $\frac{b}{a} = \tan \phi$, then this equation gives

$$\sin 2(\theta - \phi) = 0,$$

whence $2(\theta - \phi) = 0$ or π ;

$$\therefore \theta = \phi \text{ or } \phi + \frac{1}{2}\pi.$$

Taking $\theta = \phi$, the second of the given equations gives

$$a \sin^3 \phi + b \cos^3 \phi = (3ab)^{\frac{1}{2}};$$

$$\therefore a \frac{\left(\frac{b}{a}\right)^3}{\left(1 + \frac{b^2}{a^2}\right)^{\frac{3}{2}}} + b \frac{1}{\left(1 + \frac{b^2}{a^2}\right)^{\frac{3}{2}}} = (3ab)^{\frac{1}{2}};$$

$$\therefore \frac{ab^3 + ba^3}{(a^2 + b^2)^{\frac{3}{2}}} = (3ab)^{\frac{1}{2}},$$

$$\text{or } \frac{ab}{(a^2 + b^2)^{\frac{1}{2}}} = (3ab)^{\frac{1}{2}};$$

$$\therefore ab = \frac{1}{3}(a^2 + b^2),$$

$$\text{and } \frac{a^2}{b^2} - 3 \frac{a}{b} + 1 = 0;$$

$$\therefore \frac{a}{b} = \frac{3}{2} \pm \frac{1}{2}(-5)^{\frac{1}{2}} \dots\dots\dots (1).$$

Again, taking $\theta = \phi + \frac{1}{2}\pi$, the second of the given equations gives

$$a \cos^3 \phi - b \sin^3 \phi = (3ab)^{\frac{1}{2}},$$

$$\text{or } \frac{a}{\left(1 + \frac{b^2}{a^2}\right)^{\frac{3}{2}}} - \frac{b \left(\frac{b}{a}\right)^3}{\left(1 + \frac{b^2}{a^2}\right)^{\frac{3}{2}}} = (3ab)^{\frac{1}{2}},$$

$$\frac{a^4 - b^4}{(a^2 + b^2)^{\frac{3}{2}}} = (3ab)^{\frac{1}{2}};$$

$$\therefore a^2 - b^2 = (a^2 + b^2)^{\frac{1}{2}} (3ab)^{\frac{1}{2}},$$

$$a^4 - 2a^2b^2 + b^4 = 3ab(a^2 + b^2),$$

$$(a^2 + b^2)^2 - 4a^2b^2 - 3ab(a^2 + b^2) = 0;$$

$$\therefore (a^2 + b^2 - 4ab)(a^2 + b^2 + ab) = 0;$$

$$\text{therefore, first, } a^2 + b^2 - 4ab = 0,$$

$$\text{or } \left(\frac{a}{b}\right)^2 - 4\left(\frac{a}{b}\right) + 1 = 0,$$

$$\text{and } \frac{a}{b} = 2 \pm (3)^{\frac{1}{2}} \dots\dots\dots (2):$$

or, secondly,

$$a^2 + b^2 + ab = 0,$$

and

$$\left(\frac{a}{b}\right)^2 + \frac{a}{b} + 1 = 0;$$

$$\therefore \frac{a}{b} = -\frac{1}{2} \pm \frac{1}{2}(-3)^{\frac{1}{2}} \dots\dots\dots(3).$$

The six values given by (1), (2), and (3), are all the values which $\frac{a}{b}$ admits of.

1850.

1. If through the angles of a square four straight lines be drawn externally, making the same angle α with the successive sides, so as to form another square, find its area.

Let a be the side of the interior square: then the area of the exterior square = interior square + four triangles each equal to $\frac{1}{2}a^2 \sin \alpha \cos \alpha$, or $= a^2 + 4 \times \frac{1}{4}a^2 \sin 2\alpha = a^2(1 + \sin 2\alpha)$.

2. Shew that

$$2 \tan^{-1} \{ \tan^{\frac{1}{2}}(45^\circ - \alpha) \tan^{\frac{1}{2}}\beta \} = \cos^{-1} \left(\frac{\tan \alpha + \cos \beta}{1 + \tan \alpha \cos \beta} \right),$$

if α be $< 45^\circ$.

In general $\cos(2 \tan^{-1} x) = 2 \cos^2(\tan^{-1} x) - 1$

$$= \frac{2}{1 + x^2} - 1$$

$$= \frac{1 - x^2}{1 + x^2}.$$

Let $x = \tan^{\frac{1}{2}}(45^\circ - \alpha) \tan^{\frac{1}{2}}\beta$, then

$$\cos(2 \tan^{-1} x) = \frac{1 - \tan(45^\circ - \alpha) \tan^{\frac{1}{2}}\beta}{1 + \tan(45^\circ - \alpha) \tan^{\frac{1}{2}}\beta}$$

$$= \frac{1 - \frac{1 - \tan \alpha}{1 + \tan \alpha} \frac{1 - \cos \beta}{1 + \cos \beta}}{1 + \frac{1 - \tan \alpha}{1 + \tan \alpha} \frac{1 - \cos \beta}{1 + \cos \beta}}$$

$$= \frac{\tan \alpha + \cos \beta}{1 + \tan \alpha \cos \beta};$$

$$\therefore 2 \tan^{-1} \{ \tan^{\frac{1}{2}}(45^\circ - \alpha) \tan^{\frac{1}{2}}\beta \} = \cos^{-1} \left(\frac{\tan \alpha + \cos \beta}{1 + \tan \alpha \cos \beta} \right).$$

If α were $> 45^\circ$, the given expressions would become imaginary.

3. Draw AB and AC (fig. 58) at right angles to one another, and make AB equal to twice AC ; produce CA to D until CD is equal to CB : prove that BD will be the side of a regular pentagon inscribed in a circle, of which AB is the radius.

Also, if with centre D and radius DA we describe a circle AEF , of which ADF is a diameter, and make AE equal to AB , then FE will be the side of a regular pentagon circumscribing a circle, of which AC is the radius.

(α) Let $AC = a$, then $AB = 2a$;

$$\therefore CD = CB = (AB^2 + AC^2)^{\frac{1}{2}} = 5^{\frac{1}{2}}a;$$

$$\therefore AD = (5^{\frac{1}{2}} - 1)a,$$

$$\begin{aligned} BD &= (AD^2 + AB^2)^{\frac{1}{2}} \\ &= (6 - 2.5^{\frac{1}{2}} + 4)^{\frac{1}{2}}a \\ &= (10 - 2.5^{\frac{1}{2}})^{\frac{1}{2}}a \\ &= 4a \frac{(10 - 2.5^{\frac{1}{2}})^{\frac{1}{2}}}{4} \\ &= 2 \sin \frac{1}{3}\pi \cdot 2a \\ &= 2 \sin \frac{1}{3}\pi \cdot AD; \end{aligned}$$

therefore BD is the side of a regular pentagon inscribed in a circle, of which AD is the radius.

$$\begin{aligned} (\beta) \text{ Again, } FE^2 &= AF^2 - AE^2 \\ &= (2 \cdot AD)^2 - AB^2 \\ &= 4(6 - 2.5^{\frac{1}{2}})a^2 - 4a^2 \\ &= (20 - 8.5^{\frac{1}{2}})a^2 \\ &= 2^2(5 - 2.5^{\frac{1}{2}})(a)^2 \\ &= (2 \tan \frac{1}{3}\pi)^2 AC^2; \end{aligned}$$

therefore FE is the side of a regular pentagon circumscribed about a circle, of which AC is the radius.

4. If a, b, c , be the sides of the triangle ABC , p, q, r lines bisecting the angles drawn to the opposite sides, and p', q', r' these lines produced to meet the circle which circumscribes the triangle; shew that

$$\frac{\cos \frac{1}{2}A}{p} + \frac{\cos \frac{1}{2}B}{q} + \frac{\cos \frac{1}{2}C}{r} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

$$p' \cos \frac{1}{2}A + q' \cos \frac{1}{2}B + r' \cos \frac{1}{2}C = a + b + c.$$

(α) Let AD (fig. 59) be the line bisecting the angle BAC , D, D' the points in which it meets BC and the circumscribing circle respectively.

Draw DG, DH , perpendicular respectively to AB, AC ; then $DG = DH = p \sin \frac{1}{2}A$,

and $DG \cdot AB + DH \cdot AC = 2$ (area of triangle)

$$= AB \cdot AC \sin A;$$

$$\therefore p \sin \frac{1}{2}A (b + c) = bc \sin A$$

$$= bc 2 \sin \frac{1}{2}A \cos \frac{1}{2}A;$$

$$\therefore \frac{2 \cos \frac{1}{2}A}{p} = \frac{1}{b} + \frac{1}{c}.$$

Similarly, $\frac{2 \cos \frac{1}{2}B}{q} = \frac{1}{c} + \frac{1}{a},$

$$\frac{2 \cos \frac{1}{2}C}{r} = \frac{1}{a} + \frac{1}{b};$$

$$\therefore \frac{\cos \frac{1}{2}A}{p} + \frac{\cos \frac{1}{2}B}{q} + \frac{\cos \frac{1}{2}C}{r} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

(β) Again, join BD', CD' : these lines will be equal to one another, since they subtend equal angles at the circumference.

But $BD'^2 = c^2 + p'^2 - 2p'c \cos \frac{1}{2}A,$

$$CD'^2 = b^2 + p'^2 - 2p'b \cos \frac{1}{2}A;$$

$$\therefore c^2 - 2p'c \cos \frac{1}{2}A = b^2 - 2p'b \cos \frac{1}{2}A;$$

$$\therefore 2p' \cos \frac{1}{2}A = b + c.$$

Similarly, $2q' \cos \frac{1}{2}B = c + a,$

$$2r' \cos \frac{1}{2}C = a + b;$$

$$\therefore p' \cos \frac{1}{2}A + q' \cos \frac{1}{2}B + r' \cos \frac{1}{2}C = a + b + c.$$

1851.

1. If ABC be a triangle right-angled at C , E the point in which the inscribed circle touches BC , and F the point in which the circle drawn to touch AB and the sides CA , CB produced meets CA : shew that if EF be joined, the triangle FEC is half the triangle ABC .

Let r , r' be the radii of the inscribed and escribed circles respectively; then, since C is a right angle,

$$CE = r \text{ and } CF = r', \text{ and}$$

$$\text{triangle } FEC = \frac{1}{2}rr'$$

$$= \frac{1}{2} \left\{ \frac{(s-a)(s-b)(s-c)}{s} \right\}^{\frac{1}{2}} \left\{ \frac{s(s-a)(s-b)}{s-c} \right\}^{\frac{1}{2}},$$

adopting the usual notation,

$$= \frac{1}{2}(s-a)(s-b)$$

$$= \frac{1}{8}(b+c-a)(a+c-b)$$

$$= \frac{1}{8}\{c^2 - (a-b)^2\}$$

$$= \frac{1}{8}\{a^2 + b^2 - (a-b)^2\} \quad \because c^2 = a^2 + b^2,$$

$$= \frac{1}{4}ab$$

$$= \frac{1}{2} \text{ the triangle } ABC.$$

2. Shew that $\sin\beta \sin\gamma \sin(\gamma - \beta) + \sin\gamma \sin\alpha \sin(\alpha - \gamma) + \sin\alpha \sin\beta \sin(\beta - \alpha) + \sin(\gamma - \beta) \sin(\alpha - \gamma) \sin(\beta - \alpha) = 0$.

We have, in general,

$$\sin A \sin B \sin C$$

$$= \frac{1}{2} \sin A \{\cos(B-C) - \cos(B+C)\}$$

$$= \frac{1}{4} \{\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) - \sin(A+B+C)\}.$$

Hence, if $A = \beta$, $B = \gamma$, $C = \gamma - \beta$,

$$\sin\beta \sin\gamma \sin(\gamma - \beta) = \frac{1}{4} \{\sin 2(\gamma - \beta) + \sin 2\beta - \sin 2\gamma\}.$$

Similarly,

$$\sin\gamma \sin\alpha \sin(\alpha - \gamma) = \frac{1}{4} \{\sin 2(\alpha - \gamma) + \sin 2\gamma - \sin 2\alpha\},$$

$$\sin\alpha \sin\beta \sin(\beta - \alpha) = \frac{1}{4} \{\sin 2(\beta - \alpha) + \sin 2\alpha - \sin 2\beta\}.$$

Again, if $A = \gamma - \beta$, $B = \alpha - \gamma$, $C = \beta - \alpha$,

$$\begin{aligned} & \sin(\gamma - \beta) \sin(\alpha - \gamma) \sin(\beta - \alpha) \\ &= \frac{1}{4} \{ \sin 2(\beta - \gamma) + \sin 2(\gamma - \alpha) + \sin 2(\alpha - \beta) \}; \end{aligned}$$

therefore, adding these equations, we get

$$\begin{aligned} & \sin \beta \sin \gamma \sin(\gamma - \beta) + \sin \gamma \sin \alpha \sin(\alpha - \gamma) + \sin \alpha \sin \beta \sin(\beta - \alpha) \\ &+ \sin(\gamma - \beta) \sin(\alpha - \gamma) \sin(\beta - \alpha) = 0. \end{aligned}$$

3. The equation $\sin x = 0$ has not any imaginary roots.

We have
$$- \frac{1}{2} 2 \sin x = e^{-\frac{1}{2}x} - e^{-\frac{1}{2}x}.$$

Now, every imaginary quantity may be expressed under the form $\alpha + \frac{1}{2}\beta$. Substituting, then, this quantity for x , we get

$$\begin{aligned} \varepsilon^{-\frac{1}{2}x} - \varepsilon^{-\frac{1}{2}x} &= \varepsilon^{-\frac{1}{2}\alpha} \varepsilon^{-\beta} - \varepsilon^{-\frac{1}{2}\alpha} \varepsilon^{\beta} \\ &= \cos \alpha (\varepsilon^{-\beta} - \varepsilon^{\beta}) + \frac{1}{2} \sin \alpha (\varepsilon^{-\beta} + \varepsilon^{\beta}); \end{aligned}$$

therefore, if $\sin x = 0$, we must have

$$\begin{aligned} \cos \alpha (\varepsilon^{-\beta} - \varepsilon^{\beta}) &= 0, \\ \sin \alpha (\varepsilon^{-\beta} + \varepsilon^{\beta}) &= 0. \end{aligned}$$

These require, either that

$$\cos \alpha = 0 \quad \text{and} \quad \varepsilon^{-\beta} + \varepsilon^{\beta} = 0,$$

which cannot be satisfied by any real value of β ; or that

$$\sin \alpha = 0, \quad \text{and} \quad \varepsilon^{-\beta} - \varepsilon^{\beta} = 0,$$

which can only be satisfied by $\beta = 0$, shewing that $x = \alpha$, a real quantity: whence the equation $\sin x = 0$ has not any imaginary roots.

4. If the cosines of the angles A , B , C , of a plane triangle be in arithmetical progression, shew that $s - a$, $s - b$, $s - c$, will be in harmonic progression, s being the semi-sum of the sides.

We have

$$\cos A = 1 - 2 \sin^2 \frac{1}{2}A, \quad \cos B = 1 - 2 \sin^2 \frac{1}{2}B, \quad \cos C = 1 - 2 \sin^2 \frac{1}{2}C;$$

therefore, if $\cos A$, $\cos B$, $\cos C$, are in arithmetical progression, $\sin^2 \frac{1}{2}A$, $\sin^2 \frac{1}{2}B$, $\sin^2 \frac{1}{2}C$, are so;

$$\therefore 2 \sin^2 \frac{1}{2} B = \sin^2 \frac{1}{2} C + \sin^2 \frac{1}{2} A,$$

or
$$2 \frac{(s-a)(s-c)}{ac} = \frac{(s-a)(s-b)}{ab} + \frac{(s-c)(s-b)}{cb};$$

$$\therefore \frac{2b}{s-b} = \frac{c}{s-c} + \frac{a}{s-a},$$

or
$$\frac{2\{s-(s-b)\}}{s-b} = \frac{s-(s-c)}{s-c} + \frac{s-(s-a)}{s-a};$$

$$\therefore \frac{2s}{s-b} = \frac{s}{s-c} + \frac{s}{s-a},$$

$$\therefore \frac{2}{s-b} = \frac{1}{s-c} + \frac{1}{s-a};$$

whence $\frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}$, are in arithmetical progression;

$\therefore s-a, s-b, s-c$, are in harmonical progression.

SPHERICAL TRIGONOMETRY.

1848.

1. IN a right-angled spherical triangle, shew that

$$\sin a \tan \frac{1}{2}A - \sin b \tan \frac{1}{2}B = \sin(a - b) :$$

shew also that if E be the spherical excess,

$$\sin \frac{1}{2}E = \frac{\sin \frac{1}{2}a \sin \frac{1}{2}b}{\cos \frac{1}{2}c}; \quad \cos \frac{1}{2}E = \frac{\cos \frac{1}{2}a \cos \frac{1}{2}b}{\cos \frac{1}{2}c}.$$

(α). We have in general

$$\tan \frac{1}{2}A = \frac{1 - \cos A}{\sin A},$$

$$\text{and } \sin A = \frac{\sin C}{\sin c} \sin a = \frac{\sin a}{\sin c},$$

since C is a right angle;

$$\begin{aligned} \therefore \sin a \tan \frac{1}{2}A &= \sin c (1 - \cos A), \\ &= \sin c (1 - \tan b \cot c) \text{ by Napier's rules,} \\ &= \sin c - \tan b \cos c : \end{aligned}$$

similarly $\sin b \tan \frac{1}{2}B = \sin c - \tan a \cos c$;

$$\begin{aligned} \therefore \sin a \tan \frac{1}{2}A - \sin b \tan \frac{1}{2}B &= \cos c (\tan a - \tan b), \\ &= \frac{\cos c}{\cos a \cos b} \sin(a - b). \end{aligned}$$

But by Napier's rules, $\cos c = \cos a \cos b$,

$$\therefore \sin a \tan \frac{1}{2}A - \sin b \tan \frac{1}{2}B = \sin(a - b).$$

(β). Again,

$$\begin{aligned} \sin^2 \frac{1}{2}a \sin^2 \frac{1}{2}b + \cos^2 \frac{1}{2}a \cos^2 \frac{1}{2}b &= \frac{1}{4} \{ (1 - \cos a)(1 - \cos b) \\ &\quad + (1 + \cos a)(1 + \cos b) \}, \\ &= \frac{1}{2} (1 + \cos a \cos b), \\ &= \frac{1}{2} (1 + \cos c) \text{ by Napier's rules,} \\ &= \cos^2 \frac{1}{2}c; \end{aligned}$$

$$\therefore \left(\frac{\sin \frac{1}{2}a \sin \frac{1}{2}b}{\cos \frac{1}{2}c} \right)^2 + \left(\frac{\cos \frac{1}{2}a \cos \frac{1}{2}b}{\cos \frac{1}{2}c} \right)^2 = \sin^2 \frac{1}{2}E + \cos^2 \frac{1}{2}E \dots (1):$$

and in any spherical triangle

$$\tan \frac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{1}{2}C:$$

also $A+B=180^\circ+E-C$, hence

$$\cot \frac{1}{2}(C-E) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{1}{2}C;$$

therefore, since C is a right angle,

$$\frac{1 + \tan \frac{1}{2}E}{1 - \tan \frac{1}{2}E} = \frac{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b}{1 - \tan \frac{1}{2}a \tan \frac{1}{2}b},$$

$$\therefore \tan \frac{1}{2}E = \tan \frac{1}{2}a \tan \frac{1}{2}b,$$

$$\therefore \frac{\sin \frac{1}{2}E}{\cos \frac{1}{2}E} = \frac{\sin \frac{1}{2}a \sin \frac{1}{2}b}{\cos \frac{1}{2}c} \div \frac{\cos \frac{1}{2}a \cos \frac{1}{2}b}{\cos \frac{1}{2}c},$$

therefore, by (1), $\sin \frac{1}{2}E = \frac{\sin \frac{1}{2}a \sin \frac{1}{2}b}{\cos \frac{1}{2}c},$

$$\cos \frac{1}{2}E = \frac{\cos \frac{1}{2}a \cos \frac{1}{2}b}{\cos \frac{1}{2}c},$$

the required formulæ.

2. If three small circles be inscribed in a spherical triangle, having each of its angles 120° , so that each touches the other two as well as two sides of the triangle, prove that the radius of each of the small circles $= 30^\circ$, and that the centres of the three circles coincide with the angular points of the polar triangle.

Let ABC (fig. 60) be the triangle, draw the great circle AD to D the bisection of BC ; let O be the centre of one of the circles, through O draw the great circle EOF perpendicular to BC and intersecting AD in E , FE will be a quadrant; also join BO by a great circle BOH , BOH will bisect the angle ABC ; draw OG the great circle perpendicular to AD . OG and OF will each be equal to r the radius of the small circles: let $BC = 2a$, $BF = x$, $FD = y$.

Then, in the right-angled triangle ABD ,

$$\cos ABD = \tan BD \cot AB;$$

or, since $\angle ABD = 120^\circ$,

$$-\frac{1}{2} = \tan a \frac{1 - \tan^2 a}{2 \tan a},$$

$$= \frac{1 - \tan^2 a}{2},$$

$$\therefore \tan^2 a = 2,$$

$$\text{and } \cos a = \frac{1}{3^{\frac{1}{2}}}.$$

Again, in the right-angled triangle OBF ,

$$\sin BF = \cot OBF \cdot \tan OF,$$

$$\text{or } \sin x = \frac{1}{3^{\frac{1}{2}}} \tan r,$$

and in the right-angled triangle EOG , $OE = 90^\circ - r$, and $\angle OEG = FD = y$, and

$$\sin OG = \sin OE \sin OEG,$$

$$\text{or } \sin r = \cos r \sin y,$$

$$\therefore \sin y = \tan r;$$

$$\text{and } \cos a = \cos(x + y),$$

$$= (1 - \sin^2 x)^{\frac{1}{2}} (1 - \sin^2 y)^{\frac{1}{2}} - \sin x \sin y,$$

$$\text{or } = (1 - \frac{1}{3} \tan^2 r)^{\frac{1}{2}} (1 - \tan^2 r)^{\frac{1}{2}} - \frac{1}{3^{\frac{1}{2}}} \tan^2 r;$$

$$\therefore \cos^2 a + \frac{2}{3^{\frac{1}{2}}} \cos a \tan^2 r + \frac{1}{3} \tan^4 r = 1 - \frac{4}{3} \tan^2 r + \frac{1}{3} \tan^4 r,$$

$$\text{or } \frac{1}{3} + \frac{2}{3} \tan^2 r = 1 - \frac{4}{3} \tan^2 r,$$

$$\therefore \tan^2 r = \frac{1}{3},$$

$$\text{and } r = 30^\circ.$$

Again, let $BOH = b$, $BO = z$; then, in the right-angled triangle BHC ,

$$\cos BC = \cos BH \cdot \cos HC,$$

$$\text{or } \cos 2a = \cos b \cos a,$$

$$\therefore 2 \cos^2 a - 1 = \cos a \cos b,$$

$$\text{or } -\frac{1}{3} = \frac{1}{3^{\frac{1}{2}}} \cos b, \quad \therefore \cos a = \frac{1}{3^{\frac{1}{2}}},$$

$$\text{and } \cos b = -\frac{1}{3^{\frac{1}{2}}};$$

also in the right-angled triangle OFB ,

$$\sin OF = \sin OFB \cdot \sin OB,$$

$$\text{or } \frac{1}{2} = \frac{3^{\frac{1}{2}}}{2} \sin z;$$

$$\therefore \sin z = \frac{1}{3^{\frac{1}{2}}} = -\cos b,$$

$$\therefore b = 90^\circ + z,$$

and OH is a quadrant.

Again, joining OC , we have

$$\begin{aligned} \cos OC &= \cos OB \cos BC + \sin OB \sin OC \cos OBC, \\ &= \cos z \cos 2a + \sin z \sin 2a \cos 60^\circ, \\ &= -\frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}}} \cdot \frac{1}{3} + \frac{1}{3^{\frac{1}{2}}} \frac{2 \cdot 2^{\frac{1}{2}}}{3} \frac{1}{2}, \\ &= 0, \end{aligned}$$

and OC is a quadrant, so is also OH ; therefore O is the pole of AC : similarly, if O' , O'' be the other centres, O' is the pole of CB , and O'' of BA ; therefore $OO'O''$ is the polar triangle of ABC .

1849.

1. If P be the perimeter of a spherical triangle, of which the angles are A , B , C , and the spherical excess E , prove that

$$\sin \frac{1}{2}P = \frac{\{\sin \frac{1}{2}E \sin(A - \frac{1}{2}E) \sin(B - \frac{1}{2}E) \sin(C - \frac{1}{2}E)\}^{\frac{1}{2}}}{2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}.$$

By the expression for the sine of a side of a spherical triangle in terms of the angles,

$$\begin{aligned} \{\sin \frac{1}{2}E \sin(A - \frac{1}{2}E) \sin(B - \frac{1}{2}E) \sin(C - \frac{1}{2}E)\}^{\frac{1}{2}} &= \frac{1}{2} \sin a \sin B \sin C, \\ &= \frac{1}{2} \sin b \sin C \sin A = \frac{1}{2} \sin c \sin A \sin B, \\ &= \frac{1}{2} (\sin a \sin b \sin c)^{\frac{1}{2}} (\sin A \sin B \sin C)^{\frac{3}{2}} \dots \dots \dots (1). \end{aligned}$$

Again,

$$\cos \frac{1}{2}A = \left\{ \frac{\sin \frac{1}{2}P \sin(\frac{1}{2}P - a)}{\sin b \sin c} \right\}^{\frac{1}{2}} \quad \sin \frac{1}{2}A = \left\{ \frac{\sin(\frac{1}{2}P - b) \sin(\frac{1}{2}P - c)}{\sin b \sin c} \right\}^{\frac{1}{2}},$$

with similar expressions for the cosines and sines of $\frac{1}{2}B$ and $\frac{1}{2}C$;

$$\therefore \frac{\cos^2 \frac{1}{2}A \cos^2 \frac{1}{2}B \cos^2 \frac{1}{2}C}{\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C} = \frac{\sin^3 \frac{1}{2}P}{\sin a \sin b \sin c},$$

$$\begin{aligned}
\therefore \frac{\sin^3 \frac{1}{2} P}{\sin a \sin b \sin c} &= \frac{(\sin^2 \frac{1}{2} A \cos^2 \frac{1}{2} A)(\sin^2 \frac{1}{2} B \cos^2 \frac{1}{2} B)(\sin^2 \frac{1}{2} C \cos^2 \frac{1}{2} C)}{\sin^3 \frac{1}{2} A \sin^3 \frac{1}{2} B \sin^3 \frac{1}{2} C}, \\
&= \frac{\sin^2 A \sin^2 B \sin^2 C}{64 \sin^3 \frac{1}{2} A \sin^3 \frac{1}{2} B \sin^3 \frac{1}{2} C}; \\
\therefore \sin \frac{1}{2} P &= \frac{1}{4} (\sin a \sin b \sin c)^{\frac{1}{3}} \frac{(\sin A \sin B \sin C)^{\frac{2}{3}}}{\sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C}, \\
&= \frac{\{\sin E \sin (A - \frac{1}{2} E) \sin (B - \frac{1}{2} E) \sin (C - \frac{1}{2} E)\}^{\frac{1}{2}}}{2 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C},
\end{aligned}$$

by (1); the required formula.

1851.

1. If ABC be a spherical triangle, right-angled at C , and $\cos A = (\cos a)^2$, shew that $b + c = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, according as b and c are both less or both greater than $\frac{1}{2}\pi$.

By Napier's rules

$$\cos A = \cos a \sin B,$$

but by the conditions of the problem

$$\cos A = \cos^2 a,$$

$$\therefore \cos a = \sin B.$$

Again, by Napier's rules

$$\cos c = \cos a \cos b,$$

$$= \sin B \cos b \text{ from above,}$$

$$= \frac{\sin B}{\sin b} \sin b \cos b,$$

$$= \frac{1}{\sin c} \sin b \cos b, \text{ since } C \text{ is a right angle;}$$

$$\therefore \sin c \cos c = \sin b \cos b,$$

$$\text{or } \sin 2c = \sin 2b;$$

and b is not equal to c , as then B would be a right angle, and A would equal a , which is contrary to the equation $\cos A = \cos^2 a$; hence

$$2b + 2c = \pi \text{ or } 3\pi.$$

Now b and c are both greater or both less than $\frac{1}{2}\pi$, since $\cos A$ or $\cos^2 a = \tan b \cot c$; therefore $2b + 2c = \pi$ or 3π , according as b and c are both less or both greater than $\frac{1}{2}\pi$.

THEORY OF EQUATIONS.

1849.

1. Given $y = x\varepsilon^y$, prove that

$$\frac{y}{x} = 1 + \frac{2x}{[2]} + \frac{(3x)^2}{[3]} + \dots + \frac{(nx)^{n-1}}{[n]} + \dots$$

One root of the equation

$$y - x\varepsilon^y = 0$$

is the coefficient of $\frac{1}{y}$ in the expansion of $-\log\left(1 - \frac{x\varepsilon^y}{y}\right)$.(See Murphy's *Theory of Equations*, p. 77, Art. 62, and p. 80, Ex. 3.)

$$\text{Now } -\log\left(1 - \frac{x\varepsilon^y}{y}\right) = \frac{x\varepsilon^y}{y} + \frac{1}{2} \frac{x^2\varepsilon^{2y}}{y^2} + \dots + \frac{1}{n} \frac{x^n\varepsilon^{ny}}{y^n} + \dots$$

Expanding the exponentials, we see that the coefficient of $\frac{1}{y}$ is

$$x + \frac{2x^2}{2} + \frac{3^2x^3}{1.2.3} + \dots + \frac{n^{n-1}x^n}{1.2 \dots n} + \dots,$$

which is therefore a root of the given equation.

$$\text{Hence } \frac{y}{x} = 1 + \frac{2x}{[2]} + \frac{(3x)^2}{[3]} + \dots + \frac{(nx)^{n-1}}{[n]} + \dots$$

2. If x_1, x_2, \dots, x_n be the roots of an algebraical equation,

$$f(x) = x^n + p_1x^{n-1} + \dots + p_n = 0,$$

and no two of them be equal, then

$$\frac{1}{p_n} + \frac{1}{x_1 f'(x_1)} + \frac{1}{x_2 f'(x_2)} + \dots + \frac{1}{x_n f'(x_n)} = 0,$$

$$\frac{x_1^{m-1}}{f'(x_1)} + \frac{x_2^{m-1}}{f'(x_2)} + \dots + \frac{x_n^{m-1}}{f'(x_n)} = 0,$$

 m being a positive integer less than n .

(α). Here $f(x) = (x - x_1)(x - x_2) \dots (x - x_n)$,

we may therefore assume

$$\frac{1}{f(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \dots + \frac{A_n}{x - x_n},$$

A_1, A_2, \dots, A_n being independent of x ;

$$\begin{aligned} \therefore 1 &= A_1(x - x_2)(x - x_3) \dots (x - x_n) + A_2(x - x_3)(x - x_4) \dots (x - x_n)(x - x_1) + \dots \\ &\quad + A_n(x - x_1) \dots (x - x_{n-1}) \text{ identically,} \end{aligned}$$

Hence, putting $x = x_1$,

$$\begin{aligned} 1 &= A_1(x_1 - x_2)(x_1 - x_3) \dots (x_1 - x_n), \\ &= A_1 f'(x_1) \dots \dots \dots (1); \end{aligned}$$

$$\text{similarly } 1 = A_2 f'(x_2) \dots \dots \dots (2),$$

$$\dots = \dots \dots \dots$$

$$1 = A_n f'(x_n) \dots \dots \dots (n).$$

Hence

$$\frac{1}{f(x)} = \frac{1}{(x - x_1) f'(x_1)} + \frac{1}{(x - x_2) f'(x_2)} + \dots + \frac{1}{(x - x_n) f'(x_n)} \text{ identically.}$$

Therefore putting $x = 0$, which makes $f(x) = p_n$,

$$\frac{1}{p_n} = -\frac{1}{x_1 f'(x_1)} - \frac{1}{x_2 f'(x_2)} - \dots - \frac{1}{x_n f'(x_n)},$$

$$\therefore \frac{1}{p_n} + \frac{1}{x_1 f'(x_1)} + \frac{1}{x_2 f'(x_2)} + \dots + \frac{1}{x_n f'(x_n)} = 0.$$

$$\begin{aligned} (\beta) \text{ Again, } \frac{1}{(x - x_1) f'(x_1)} &= \frac{1}{x f'(x_1)} \left(1 - \frac{x_1}{x}\right)^{-1} \\ &= \frac{1}{x f'(x_1)} \left(1 + \frac{x_1}{x} + \frac{x_1^2}{x^2} + \dots\right), \end{aligned}$$

and similarly for the other fractions. Hence

$$\begin{aligned} \frac{1}{f(x)} &= \frac{1}{x f'(x_1)} \left(1 + \frac{x_1}{x} + \frac{x_1^2}{x^2} + \dots\right) + \frac{1}{x f'(x_2)} \left(1 + \frac{x_2}{x} + \frac{x_2^2}{x^2} + \dots\right) \\ &\quad + \dots + \frac{1}{x f'(x_n)} \left(1 + \frac{x_n}{x} + \frac{x_n^2}{x^2} + \dots\right) \dots \dots \dots (1). \end{aligned}$$

$$\begin{aligned}\text{But } \frac{1}{f(x)} &= \frac{1}{x^n + p_1 x^{n-1} + \dots + p_n} = \frac{1}{x^n \left(1 + \frac{p_1}{x} + \dots + \frac{p_n}{x^n}\right)} \\ &= \frac{1}{x^n} \left(1 - \frac{p_1}{x} + \dots\right).\end{aligned}$$

Hence, if m be a positive integer, less than n , the coefficient of $\frac{1}{x^m}$ in $\frac{1}{f(x)} = 0$; and the coefficient in the right-hand member of (1) is

$$\frac{x_1^{m-1}}{f'(x_1)} + \frac{x_2^{m-1}}{f'(x_2)} + \dots + \frac{x_n^{m-1}}{f'(x_n)}:$$

hence
$$\frac{x_1^{m-1}}{f'(x_1)} + \frac{x_2^{m-1}}{f'(x_2)} + \dots + \frac{x_n^{m-1}}{f'(x_n)} = 0.$$

1851.

1. If
$$x + \frac{1}{1-x} - \frac{1-x}{x} = -3p,$$

then
$$\left(x + \frac{\omega}{1-x} - \omega^2 \frac{1-x}{x}\right)^3 = -27(p-\omega)(p-\omega^2)^2,$$

where ω is an imaginary cube root of unity.

Call the quantities x , $\frac{1}{1-x}$ and $-\frac{1-x}{x}$, y_1 , y_2 , and y_3 ; then

$$y_1 + y_2 + y_3 = -3p \dots \dots \dots (1),$$

and $y_1 y_2 + y_2 y_3 + y_3 y_1 = \frac{x}{1-x} - \frac{1}{x} - (1-x)$

$$= \frac{1}{1-x} - 1 - \frac{1-x}{x} = 1 + x - 1$$

$$= y_2 + y_3 + y_1 - 3$$

$$= -3(p+1) \dots \dots \dots (2),$$

also $y_1 y_2 y_3 = -1 \dots \dots \dots (3).$

Again,

$$y_1^2 y_2 + y_2^2 y_3 + y_3^2 y_1 = \frac{x^2}{1-x} - \frac{1}{x(1-x)} + \frac{(1-x)^2}{x}$$

$$= -\frac{3x(1-x)}{x(1-x)} = -3 \dots \dots \dots (4),$$

$$\text{and } y_1^2 y_2 + y_2^2 y_3 + y_3^2 y_1 + y_1^2 y_3 + y_2^2 y_1 + y_3^2 y_2$$

$$= y_1 (y_1 y_2 + y_1 y_2) + \dots$$

$$= -y_1 \{3(p+1) + y_2 y_3\} + \dots \text{ by (2),}$$

$$= 9p(p+1) + 3, \text{ by (1) and (3);}$$

$$\therefore y_1^2 y_3 + y_2^2 y_1 + y_3^2 y_2 = 9p(p+1) + 6 \dots \text{ by (4) and (5).}$$

$$\text{And } (y_1 + \omega y_2 + \omega^2 y_3)^3 = y_1^3 + y_2^3 + y_3^3 + 3\omega y_1 y_2 (y_1 + \omega y_2) +$$

similar terms + $6y_1 y_2 y_3$

$$= (y_1 + y_2 + y_3)^3 + 3(\omega - 1) y_1 y_2$$

$\{y_1 + (\omega + 1) y_2\} + \text{similar terms}$

$$= (-3p)^3 - 9(\omega - 1) + 3(\omega^2 - 1)$$

$$\{9p(p+1) + 6\} \text{ by (1), (4), and (5),}$$

$$= -27p^3 + 27p^2(2\omega^2 + \omega) - 27p(2 + \omega)$$

$$+ 27\omega^2, \text{ since } \omega + \omega^2 = -1,$$

$$= -27(p - \omega)(p - \omega^2)^2$$

GEOMETRY OF TWO DIMENSIONS.

1848.

1. With two conjugate diameters of an ellipse as asymptotes a pair of conjugate hyperbolas is constructed; prove that if one hyperbola touch the ellipse the other will do so likewise; prove also that the diameters drawn through the points of contact are conjugate to each other.

Let the equation to the ellipse, referred to the conjugate diameters, be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots \dots \dots (1).$$

And to the hyperbolas

$$xy = c^2 \dots \dots \dots (2),$$

$$xy = -c^2 \dots \dots \dots (3).$$

(α) In order that (2) may touch (1), we must have

$$\frac{x^2}{a^2} - \frac{xy}{c^2} + \frac{y^2}{b^2}$$

a perfect square, in which case we shall have also

$$\frac{x^2}{a^2} + \frac{xy}{c^2} + \frac{y^2}{b^2}$$

a perfect square, and (3) will also touch (1).

(β). If the above expressions be perfect squares, we see that

$$\frac{4}{a^2 b^2} = \frac{1}{c^4};$$

$$\therefore c^2 = \frac{ab}{2},$$

$$\text{and } \frac{y}{x} = \pm \frac{b}{a};$$

and if $x'y'$ be the coordinates of the point where (1) meets (2),

$$y'^2 = \frac{bc^2}{a} = \frac{b^2}{2},$$

$$x'^2 = \frac{a^2}{2}.$$

Similarly, if $x''y''$ be the coordinates of the point, where (1) meets (3),

$$y''^2 = \frac{b^2}{2}, \quad x''^2 = \frac{a^2}{2}.$$

And if r', r'' , be the lengths of the corresponding semi-diameters, ω the angle between the axes,

$$r'^2 = x'^2 + y'^2 - 2x'y' \cos \omega = \frac{1}{2}(a^2 + b^2 - 2ab \cos \omega),$$

$$r''^2 = x''^2 + y''^2 - 2x''y'' \cos \omega = \frac{1}{2}(a^2 + b^2 + 2ab \cos \omega),$$

$$\therefore r'^2 + r''^2 = a^2 + b^2;$$

therefore r', r'' are conjugate to each other.

2. Shew that the curve which trisects the arcs of all segments of a circle described upon a given base is an hyperbola whose eccentricity = 2.

Let AB (fig. 61) be the base, a its length,

$$AC = CD = DB = r, \quad CAB = \theta,$$

we then have $r(1 + 2 \cos \theta) = a$,

or, referring the curve to A as origin and AB as axis of x ,

$$x^2 + y^2 = (a - 2x)^2;$$

$$\therefore 3x^2 - y^2 - 4ax + a^2 = 0,$$

the equation to an hyperbola, the squares of whose axes are to one another in the ratio 3 : 1, and whose eccentricity therefore = $(3 + 1)^{\frac{1}{2}} = 2$.

3. Let D be a point in the axis-minor of an ellipse whose eccentricity is e , S the focus, O the centre of curvature at the extremity of the axis-minor; with centre D and radius = $\frac{DS}{e}$

describe a circle; shew that this circle will touch the ellipse or fall entirely without it, according as D is nearer to or further from the centre than the point O .

Let C be the centre of the ellipse, and let $CD = h$, also let a, b , be the semi-axes of the ellipse, its equation will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Also $DS^2 = a^2e^2 + h^2$, so that the equation to the circle will be

$$x^2 + (y + h)^2 = a^2 + \frac{h^2}{e^2} \text{ (taking } D \text{ below } C),$$

$$\text{or } x^2 + y^2 + 2hy = a^2 + h^2 \frac{1 - e^2}{e^2}.$$

Where this meets the ellipse, we have

$$y^2 \left(\frac{1}{b^2} - \frac{1}{a^2} \right) - \frac{2hy}{a^2} + \frac{h^2}{a^2} \frac{1 - e^2}{e^2} = 0,$$

$$\text{or } \frac{e^2}{1 - e^2} y^2 - 2hy + h^2 \frac{1 - e^2}{e^2} = 0;$$

$$\therefore y = \frac{1 - e^2}{e^2} h.$$

If this value of y give a real value for x , the circle will touch the ellipse, if not, it will fall entirely without it, since its radius $\left(a^2 + \frac{h^2}{e^2} \right)^{\frac{1}{2}}$ is greater than a , and therefore, *à fortiori*, than DB' , which is less than b .

In order that the value of x may be real it is necessary that y be not greater than b , therefore

$$\frac{1 - e^2}{e^2} h < b,$$

$$\text{or } h < \frac{be^2}{1 - e^2},$$

$$< \frac{a^2}{b} - b,$$

$$< BO - BC,$$

$$< CO;$$

therefore the circle will touch the ellipse, or fall entirely without it, according as $h < \text{or} > CO$, *i.e.* as D is nearer to or further from the centre than O .

4. PSp is any focal chord of an ellipse, A the extremity of the axis-major; AP , Ap meet the directrix in two points Q , q : shew that $\angle QSq$ is a right angle.

We may prove this property for any point in the ellipse by a process exactly similar to that of Part I. *Conics*, 1848, 3; except that we have the equations

$$\sin PRS = e \sin PSR \cdot \sin PRN,$$

$$\sin QRS = e \sin QRS \sin PRN,$$

instead of those there given.

This theorem may also be proved by the method of Reciprocal Polars. (See Salmon's *Conic Sections*, chap. XIV.)

Take the polar reciprocal of the whole system with regard to the focus S . To the ellipse will correspond a circle, to the point P , p , two parallel tangents Rt , rt' , (fig. 62) variable in position. To A (or any point in the curve) will correspond a fixed tangent tt' , and to the directrix the centre S' . Hence to AP , Ap will correspond the points t , t' respectively, and to Q , q the lines St , St' . But it is easy to see that the lines St , St' are at right angles to one another; therefore the line joining the points Q , q , subtends a right angle at the focus S .

5. In the given right lines AP , AQ , (fig. 63) are taken variable points p , q , such that $Ap : pP :: Qq : qA$; prove that the locus of the point of intersection of Pq , Qp is an ellipse, which touches the given right lines in the points P , Q .

Let $AP = a$, $AQ = b$, $Ap = \alpha$, $Aq = \beta$; then the conditions of the problem give

$$\alpha : a - \alpha :: b - \beta : \beta, \quad \text{or} \quad \frac{\alpha}{a} + \frac{\beta}{b} = 1 \dots\dots\dots (1).$$

Take AP , AQ , as axes; then the equations to Pq , pQ respectively, are

$$\frac{x}{a} + \frac{y}{\beta} = 1 \dots \dots \dots (2),$$

$$\frac{x}{\alpha} + \frac{y}{b} = 1 \dots \dots \dots (3).$$

Hence
$$\frac{\frac{x}{\alpha}}{1 - \frac{y}{b}} = \frac{\frac{y}{\beta}}{1 - \frac{x}{a}} = 1;$$

whence, eliminating α , β from (1), we get

$$\frac{\frac{x}{a}}{1 - \frac{y}{b}} + \frac{\frac{y}{b}}{1 - \frac{x}{a}} = 1;$$

$$\therefore \frac{x^2}{a^2} + \frac{xy}{ab} + \frac{y^2}{b^2} - 2\frac{x}{a} - 2\frac{y}{b} + 1 = 0,$$

the equation to the locus of the intersection of Pq , pQ , which, since the square of half the coefficient of xy is less than the product of the coefficients of x^2 and y^2 , is an ellipse.

When $x = 0$, we have

$$\frac{y^2}{b^2} - 2\frac{y}{b} + 1 = 0;$$

$$\therefore y = b,$$

shewing that the ellipse touches AP in P .

From considerations of symmetry it is evident that it also touches AQ in Q .

6. A parallelogram is constructed by drawing tangents at the extremities of two conjugate diameters of an ellipse; prove that the diagonals of the parallelogram form a second system of conjugate diameters, and that the relation between the two systems is reciprocal.

Let the equation to the ellipse referred to the conjugate diameters be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots\dots\dots(1).$$

The equations to the tangents, drawn at the extremities of these diameters, are

$$\begin{aligned} x &= a, & x &= -a, \\ y &= b, & y &= -b; \end{aligned}$$

therefore the equations to the diagonals of the parallelogram thus formed, are

$$\frac{x}{a} = \frac{y}{b} \dots\dots\dots(2),$$

$$\frac{x}{a} = -\frac{y}{b} \dots\dots\dots(3).$$

At the points where (2) meets (1), we have

$$x = \pm \frac{a}{\sqrt{2}}, \quad y = \pm \frac{b}{\sqrt{2}};$$

therefore the equations to the tangents at these points are

$$\frac{x}{a} + \frac{y}{b} = \pm \sqrt{2} \dots\dots\dots(4),$$

therefore these tangents are parallel to (3). Hence the diagonals form a system of conjugate diameters.

Again, the equations to the tangents at the extremities of (3) are

$$\frac{x}{a} - \frac{y}{b} = \pm \sqrt{2} \dots\dots\dots(5),$$

and at the intersection of (4) and (5), we have either

$$x = 0, \quad \text{or} \quad y = 0,$$

shewing that the diagonals of the parallelogram, formed by the lines (4) and (5), are the first system of conjugate diameters; hence the relation between the systems is reciprocal.

7. *PSp* is any focal chord of a parabola whose vertex is *A*, prove geometrically that *AP*, *Ap* will meet the latus-rectum in two points *Q*, *q*, whose distances from the focus are equal to the ordinates of the points *p* and *P* respectively.

Draw PM , pm , (fig. 64) ordinates to the points P , p respectively. Then since SQ is parallel to PM ,

$$\therefore SQ : AS :: MP : AM,$$

$$\therefore SQ : 4AS^2 :: MP : 4AS.AM,$$

$$:: MP : MP^2;$$

$$\therefore SQ.MP = 4AS^2.$$

$$\text{Now } SP = 2AS + SM,$$

$$= 2AS + SP \cos PSM;$$

$$\therefore SP(1 - \cos PSM) = 2AS,$$

$$\therefore PM(1 - \cos PSM) = 2AS \sin PSM:$$

$$\text{similarly } pm(1 + \cos pSm) = 2AS \sin pSm;$$

$$\therefore PM.pm = 4AS^2,$$

$$= SQ.PM \text{ from above;}$$

$$\therefore pm = SQ,$$

$$\text{similarly } PM = Sq.$$

Or the distances of Q , q , from the focus are equal to the ordinates of the points p , P respectively.

8. From a given point in a conic section, draw geometrically two chords at right angles to each other which shall be in a given ratio.

The construction which we shall give depends on the property that all chords of a conic section which subtend a right angle at a given point P of the curve, intersect the normal at P in a fixed point.

Draw PK (fig. 65) the normal at P , and draw PU , PV any two chords at right angles to one another. Join UV , cutting the normal in K . Then by the property above enunciated, if PQ , PR are the required chords, QR will pass through K . Again, $\tan PQR = \frac{PR}{PQ}$ a given ratio, hence the angle PQR is known; on PK we describe a segment of a circle containing an angle equal to PQR ; let it cut the ellipse in Q . Join PQ , and draw PR at right angles to it, PQ , PR will be the required chords.

9. Determine the equation to the conic section which passes through five points whose coordinates are given; and thence shew that the equation to the conic section which passes through the five points whose coordinates are

$$1, -1; 2, 1; -2, 3; 3, 2; -1, -3,$$

$$\text{is } 61y^2 - 17xy - 65x^2 + 36y + 174x - 151 = 0.$$

Let $x_1, y_1; x_2, y_2; x_3, y_3; x_4, y_4; x_5, y_5$ be the coordinates of the five given points, which we shall call A_1, A_2, A_3, A_4, A_5 respectively. Then the conic passing through A_2, A_3, A_4, A_5 , circumscribes the quadrilateral, whose sides are $A_2A_3, A_3A_4, A_4A_5, A_5A_2$. The equations to these sides are

$$\frac{x - x_2}{x_2 - x_3} - \frac{y - y_2}{y_2 - y_3} = 0,$$

$$\frac{x - x_3}{x_3 - x_4} - \frac{y - y_3}{y_3 - y_4} = 0,$$

$$\frac{x - x_4}{x_4 - x_5} - \frac{y - y_4}{y_4 - y_5} = 0,$$

$$\frac{x - x_5}{x_5 - x_2} - \frac{y - y_5}{y_5 - y_2} = 0.$$

Now the equation to a conic, circumscribing a quadrilateral, the equations to whose sides are $u_2 = 0, u_3 = 0, u_4 = 0, u_5 = 0$ respectively, is

$$u_2 u_4 = \lambda u_3 u_5,$$

λ being an indeterminate parameter.

Hence the equation to a conic passing through A_2, A_3, A_4, A_5 , is

$$\left(\frac{x - x_2}{x_2 - x_3} - \frac{y - y_2}{y_2 - y_3} \right) \left(\frac{x - x_4}{x_4 - x_5} - \frac{y - y_4}{y_4 - y_5} \right)$$

$$= \lambda \left(\frac{x - x_3}{x_3 - x_4} - \frac{y - y_3}{y_3 - y_4} \right) \left(\frac{x - x_5}{x_5 - x_2} - \frac{y - y_5}{y_5 - y_2} \right).$$

The quantity λ is determined by the condition of the conic passing through the point $A_1(x_1, y_1)$; this gives

$$\left(\frac{x_1 - x_2}{x_2 - x_3} - \frac{y_1 - y_2}{y_2 - y_3} \right) \left(\frac{x_1 - x_4}{x_4 - x_5} - \frac{y_1 - y_4}{y_4 - y_5} \right)$$

$$= \lambda \left(\frac{x_1 - x_3}{x_3 - x_4} - \frac{y_1 - y_3}{y_3 - y_4} \right) \left(\frac{x_1 - x_5}{x_5 - x_2} - \frac{y_1 - y_5}{y_5 - y_2} \right).$$

Eliminating λ between these last two equations we get, clearing the quantities within the brackets of fractions,

$$\frac{\{(x-x_2)(y_2-y_3)-(y-y_2)(x_2-x_3)\}\{(x-x_4)(y_4-y_5)-(y-y_4)(x_4-x_5)\}}{\{(x_1-x_2)(y_2-y_3)-(y_1-y_2)(x_2-x_3)\}\{(x_1-x_4)(y_4-y_5)-(y_1-y_4)(x_4-x_5)\}} \\ = \frac{\{(x-x_3)(y_3-y_4)-(y-y_3)(x_3-x_4)\}\{(x-x_5)(y_5-y_2)-(y-y_5)(x_5-x_2)\}}{\{(x_1-x_3)(y_3-y_4)-(y_1-y_3)(x_3-x_4)\}\{(x_1-x_5)(y_5-y_2)-(y_1-y_5)(x_5-x_2)\}},$$

the equation to the required conic. The reduction of this to the symmetrical form would be very tedious, and we shall therefore leave it in the above shape.

In the numerical example

$$x_1 = 1, y_1 = -1; x_2 = 2, y_2 = 1; x_3 = -2, y_3 = 3;$$

$$x_4 = 3, y_4 = 2; x_5 = -1, y_5 = -3.$$

Hence

$$x_1 - x_2 = -1; x_1 - x_3 = 3; x_1 - x_4 = -2; x_1 - x_5 = 2,$$

$$y_1 - y_2 = -2; y_1 - y_3 = -4; y_1 - y_4 = -3; y_1 - y_5 = 2,$$

$$x_2 - x_3 = 4; x_3 - x_4 = -5; x_4 - x_5 = 4; x_5 - x_2 = -3,$$

$$y_2 - y_3 = -2; y_3 - y_4 = 1; y_4 - y_5 = 5; y_5 - y_2 = -4;$$

therefore the above equation becomes

$$\frac{\{-2(x-2)-4(y-1)\}\{5(x-3)-4(y-2)\}}{\{(-2)(-1)-4(-2)\}\{5(-2)-4(-3)\}} \\ = \frac{\{1(x+2)-(-5)(y-3)\}\{-4(x+1)-(-3)(y+3)\}}{\{1(3)-(-5)(-4)\}\{-4(2)-(-3)2\}};$$

$$\therefore \frac{(8-2x-4y)(-7+5x-4y)}{20} = \frac{(-13+x+5y)(5-4x+3y)}{34},$$

$$\therefore 17(5x-4y-7)(x+2y-4) - 5(x+5y-13)(4x-3y-5);$$

$$\therefore 65x^2 + 17xy - 61y^2 - 174x - 36y + 151 = 0,$$

$$\text{or } 61y^2 - 17xy - 65x^2 + 36y + 174x - 151 = 0,$$

is the equation to the conic passing through the five given points.

10. Two chords AB , AC are drawn from a given point A in a curve of the second order so as to contain a given angle, shew that BC will always touch a curve of the second order.

Let the equation to the given conic section, referred to A as origin, be

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey = 0 \dots\dots\dots(1),$$

$$\text{and let } \alpha x + \beta y = 1 \dots\dots\dots(2)$$

be the equation to BC , α, β being variable parameters.

At the points of intersection of (1) and (2), we have

$$Ax^2 + 2Bxy + Cy^2 + 2(Dx + Ey)(\alpha x + \beta y) = 0,$$

$$\text{or } (C + 2E\beta)y^2 + 2(B + E\alpha + D\beta)xy + (A + 2D\alpha)x^2 = 0 \dots\dots(3).$$

This may be considered as a quadratic in $\frac{y}{x}$, and if t_1, t_2 be its roots, t_1, t_2 will be the tangents of the inclinations to the axis of x of AB, AC respectively. But AB, AC include a constant angle, $\tan^{-1}m$ suppose; hence we must have

$$\frac{t_1 - t_2}{1 + t_1 t_2} = m,$$

$$\text{or } \frac{(t_1 + t_2)^2 - 4t_1 t_2}{(1 + t_1 t_2)^2} = m^2.$$

Now from (3), by the the theory of equations,

$$t_1 + t_2 = -2 \frac{B + E\alpha + D\beta}{C + 2E\beta}, \quad t_1 t_2 = \frac{A + 2D\alpha}{C + 2E\beta};$$

$$\therefore 4\{(B + E\alpha + D\beta)^2 - (A + 2D\alpha)(C + 2E\beta)\} \\ = m^2\{(A + 2D\alpha) + C + 2E\beta\}^2,$$

$$\therefore 4\{B^2 - AC + 2(BE - CD)\alpha + 2(BD - AE)\beta + E^2\alpha^2 \\ + D^2\beta^2 + 2(B + DE)\alpha\beta\}, \\ = m^2\{A + C + 2(D\alpha + E\beta)\}^2,$$

which may be written under the form

$$ax^2 + 2b\alpha\beta + c\beta^2 + 2(d\alpha + e\beta) + 1 = 0 \dots\dots\dots(4),$$

a, b, c, d, e being certain determinate functions of A, B, C, D, E , and m .

Now consider the conic section whose equation is

$$A'x^2 + 2B'xy + C'y^2 + 2(D'x + E'y) + 1 = 0 \dots\dots\dots(5).$$

Where (2) meets this, we have

$$\begin{aligned} A'x^2 + 2B'xy + C'y^2 + 2(D'x + E'y)(\alpha x + \beta y) + (\alpha x + \beta y)^2 &= 0; \\ \therefore (A' + 2D'\alpha + \alpha^2)x^2 + 2(B' + E'\alpha + D'\beta + \alpha\beta)xy \\ &\quad + (C' + 2E'\beta + \beta^2)y^2 = 0. \end{aligned}$$

In order that (2) may touch (5) the roots of this equation, considered as a quadratic in $\frac{y}{x}$, must be equal; we must therefore have

$$\begin{aligned} (B' + E'\alpha + D'\beta + \alpha\beta)^2 &= (A' + 2D'\alpha + \alpha^2)(C' + 2E'\beta + \beta^2); \\ \therefore B'^2 - A'C' + 2(B'E' - C'D')\alpha + 2(B'D' - A'E')\beta \\ &\quad + (E'^2 - C')\alpha^2 + 2(B' - D'E')\alpha\beta + (D'^2 - A')\beta^2 = 0, \end{aligned}$$

which agrees with (4) if

$$\begin{aligned} \frac{E'^2 - C'}{B'^2 - A'C'} &= a, \quad \frac{B' - D'E'}{B'^2 - A'C'} = b, \quad \frac{D'^2 - A'}{B'^2 - A'C'} = c, \\ \frac{B'E' - C'D'}{B'^2 - A'C'} &= d, \quad \frac{B'D' - A'E'}{B'^2 - A'C'} = e, \end{aligned}$$

which five conditions can be satisfied by means of the five disposable quantities A', B', C', D', E' . Hence BC always touches the conic whose equation is (5).

This theorem may also be proved by the method of reciprocal polars. For taking the polar reciprocal of the whole system with regard to A ; to the conic will correspond a parabola, and to B, C (two points the line joining which subtends a constant angle at the origin) will correspond two tangents containing a constant angle. The reciprocal theorem then is:

If two tangents be drawn to a parabola, including a constant angle, the locus of their point of intersection is a curve of the second order.*

This may be proved as follows:

* Taking the polar reciprocal of *this* system with regard to the focus of the parabola, the theorem to be proved is the following:

If a chord of a circle subtend a constant angle at a given point of the curve, it always touches a circle, which is known to be true.

Let

$$y = tx + \frac{a}{t}$$

be the equation to any tangent to a parabola. This may be written

$$t^2 - \frac{y}{x}t + \frac{a}{x} = 0 \dots \dots \dots (1).$$

This equation, considered as a quadratic in t , gives the tangents of the inclinations to the axis of the two tangents drawn to a parabola through a point (xy) . In order that these may include a given angle $\tan^{-1}m$, we must have, if t_1, t_2 be the roots of (1),

$$\frac{t_1 - t_2}{1 + t_1 t_2} = m;$$

therefore, by the theory of equations,

$$\frac{\left(\frac{y}{x}\right)^2 - \frac{4a}{x}}{\left(1 + \frac{a}{x}\right)^2} = m^2,$$

$$\text{or } y^2 - 4ax = m^2(a+x)^2,$$

the equation to the locus of xy , which is therefore a curve of the second order.

11. P, D are the extremities of two semi-conjugate diameters of an ellipse E , whose semi-axes are a, b ; upon PD describe an equilateral triangle PDR , so that the point R may fall without the ellipse; the locus of R will be an ellipse E_1 : assuming the above result, shew that if E_1 be similarly treated, as also all the successive ellipses, the axes A_x, B_x of the x^{th} ellipse E_x so described will be comprised in the formula

$$(a+b)(\cot \frac{1}{12}\pi)^{\frac{1}{2}x} \pm (a-b)(\cot \frac{1}{12}\pi)^{\frac{1}{2}x}.$$

In the figure (66), let CP, CD represent the equal semi-conjugate diameters of the ellipse E , and let D' be the other extremity of the diameter through D . Join PD, PD' ; on them describe the equilateral triangles $PDR, PD'R'$; then R, R' will be the extremities of the axes of E_1 . Join CR, CR' . Then

$$CR' = \frac{1}{2}A_1, \quad CR = \frac{1}{2}B_1;$$

also $\frac{PV'}{CV'} = \tan PCR' = \frac{b}{a}$, and $PV' \cdot CV' = \frac{1}{2}ab$;

$$\therefore CV' = \frac{1}{2^{\frac{1}{2}}}a, \text{ and } PV' = \frac{1}{2^{\frac{1}{2}}}b.$$

$$\begin{aligned} \text{And } CR' &= CV' + V'R', \\ &= \frac{a}{2^{\frac{1}{2}}} + PD' \cos \frac{\pi}{6}, \\ &= \frac{a}{2^{\frac{1}{2}}} + 2^{\frac{1}{2}}b \cos \frac{\pi}{6}. \end{aligned}$$

$$\text{Similarly } CR = \frac{b}{2^{\frac{1}{2}}} + 2^{\frac{1}{2}}a \cos \frac{\pi}{6},$$

$$\frac{1}{2}(B_1 + A_1) = (a + b) \left(2^{\frac{1}{2}} \cos \frac{\pi}{6} + \frac{1}{2^{\frac{1}{2}}} \right),$$

$$\frac{1}{2}(B_1 - A_1) = (a - b) \left(2^{\frac{1}{2}} \cos \frac{\pi}{6} - \frac{1}{2^{\frac{1}{2}}} \right).$$

$$\begin{aligned} \text{Now } 2^{\frac{1}{2}} \cos \frac{\pi}{6} + \frac{1}{2^{\frac{1}{2}}} &= \frac{3^{\frac{1}{2}} + 1}{2^{\frac{1}{2}}} = \frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}} - 1} \\ &= \left(\frac{3^{\frac{1}{2}} + 1}{3^{\frac{1}{2}} - 1} \right)^{\frac{1}{2}} = \left(\frac{2 + 3^{\frac{1}{2}}}{2 - 3^{\frac{1}{2}}} \right)^{\frac{1}{4}} = \left(\frac{1 + \cos \frac{1}{6}\pi}{1 - \cos \frac{1}{6}\pi} \right)^{\frac{1}{4}}, \\ &= \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}}. \end{aligned}$$

Similarly it may be shewn that

$$2^{\frac{1}{2}} \cos \frac{\pi}{6} - \frac{1}{2^{\frac{1}{2}}} = \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}};$$

$$\therefore \frac{1}{2}(B_1 + A_1) = (a + b) \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}},$$

$$\frac{1}{2}(B_1 - A_1) = (a - b) \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}},$$

a formula connecting the axes of any two successive ellipses ;

$$\therefore B_1 = (a + b) \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}} + (a - b) \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}},$$

$$A_1 = (a + b) \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}} - (a - b) \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}}.$$

Now assume

$$B_x = (a+b) \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}x} - {}^{-x}(a-b) \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}x},$$

$$A_x = (a+b) \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}x} + {}^{-x}(a-b) \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}x};$$

then, by the formula already given,

$$\begin{aligned} B_{x+1} &= \frac{A_x + B_x}{2} \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}} + \frac{A_x - B_x}{2} \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}}, \\ &= (a+b) \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}(x+1)} + {}^{-x+1}(a-b) \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}(x+1)}; \end{aligned}$$

$$\begin{aligned} \text{similarly } A_{x+1} &= \frac{A_x + B_x}{2} \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}} + \frac{A_x - B_x}{2} \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}}, \\ &= (a+b) \left(\cot \frac{\pi}{12} \right)^{\frac{1}{2}(x+1)} + {}^{-x+1}(a-b) \left(\tan \frac{\pi}{12} \right)^{\frac{1}{2}(x+1)}. \end{aligned}$$

If then the assumed form hold for E_x it is proved to hold for E_{x+1} . But it has been shewn to hold for E_1 , therefore it holds universally.

1849.

A is the origin (fig. 67), B a point in the axis of y , BQ a line parallel to the axis of x ; in AQ (produced if necessary) P is taken such that its ordinate is equal to BQ : shew that the locus of P is a parabola.

Let $AB = a$, $AP = r$, $BAP = \frac{1}{2}\pi - \theta$, then the ordinate of $P = r \sin \theta$, also $BQ = a \cot \theta$;

$$\therefore r \sin \theta = a \cot \theta,$$

$$\text{or } r^2 \sin^2 \theta = ar \cos \theta;$$

therefore, putting $r \sin \theta = y$, $r \cos \theta = x$,

$$y^2 = ax,$$

shewing that the locus of P is a parabola.

2. If from points of the curve $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$, tangents be drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the chords of contact will be normal to the ellipse.

If ξ, η be the current coordinates of the normal to the ellipse at the point (xy) , its equation will be

$$\frac{\xi - x}{\frac{x}{a^2}} = \frac{\eta - y}{\frac{y}{b^2}},$$

$$\text{or } \frac{a^2}{a^2 - b^2} \frac{\xi}{x} + \frac{b^2}{b^2 - a^2} \frac{\eta}{y} = 1.$$

If x_1, y_1 be the coordinates of the pole of this line with respect to the given ellipse, we have

$$x_1 = \frac{a^4}{a^2 - b^2} \frac{1}{x},$$

$$y_1 = \frac{b^4}{b^2 - a^2} \frac{1}{y}.$$

Eliminating x, y between these two equations and the equation to the ellipse, we get

$$\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2,$$

the equation to the locus of the pole of the normals to the ellipse.

Hence if from points of this curve, tangents be drawn to the given ellipse, the chords of contact will be normal to the ellipse.

3. An oblique cone stands on a circular base; prove that one of the axes of the section made by a plane passing through the centre of the base and perpendicular to the axis, is a mean proportional between the other axis and $D \sec \alpha$, where D is the diameter of the base, and α the angle between the axis and a normal to the base.

Let MPM' (fig. 68) be the section through O the centre of the base and perpendicular to CO the axis of the cone; let the section intersect the circular section RPR' in the line NP .

Let a and b be the axes of the section, then

$$\frac{b^2}{a^2} = \frac{NP^2}{MN.NM'} = \frac{RN.NR'}{MN.NM'}.$$

Let $\angle CAO = \beta$, $CBO = \gamma$; also the $\angle MOA$ is given equal to α . Hence

$$\frac{RN}{MN} = \frac{\sin(\beta + \alpha)}{\sin \beta}, \quad \frac{NR'}{NM'} = \frac{\sin(\beta' - \alpha)}{\sin \beta'};$$

$$\therefore \frac{b^2}{a^2} = \frac{\sin(\beta + \alpha)}{\sin \beta} \cdot \frac{\sin(\beta' - \alpha)}{\sin \beta'} \dots\dots\dots (1),$$

$$\text{and } a = MM' = MO + OM' = \frac{D}{2} \left\{ \frac{\sin \beta}{\sin(\beta + \alpha)} + \frac{\sin \beta'}{\sin(\beta' - \alpha)} \right\} \dots (2).$$

Again, since $AO = OB$,

$$\frac{\sin ACO}{\sin CAO} = \frac{\sin BCO}{\sin OBC},$$

$$\text{or } \frac{\cos(\beta + \alpha)}{\sin \beta} = \frac{\cos(\beta' - \alpha)}{\sin \beta'} = p \text{ suppose};$$

$$\therefore \cot \beta = \frac{p + \sin \alpha}{\cos \alpha}, \quad \cot \beta' = \frac{p - \sin \alpha}{\cos \alpha}.$$

Substituting in equations (1) and (2), we have

$$\frac{b^2}{a^2} = (\cos \alpha + \cot \beta \sin \alpha)(\cos \alpha - \cot \beta' \sin \alpha),$$

$$= \frac{1}{\cos^2 \alpha} (1 + p \sin \alpha)(1 - p \sin \alpha),$$

$$= \frac{1 - p^2 \sin^2 \alpha}{\cos^2 \alpha};$$

$$\text{and } a = \frac{D}{2} \left(\frac{\cos \alpha}{1 + p \sin \alpha} + \frac{\cos \alpha}{1 - p \sin \alpha} \right),$$

$$= \frac{D \cos \alpha}{1 - p^2 \sin^2 \alpha};$$

$$\therefore = \frac{a^2}{b^2} D \sec \alpha,$$

$$\text{or } b^2 = aD \sec \alpha,$$

and b is a mean proportional between a and $D \sec \alpha$.

4. Let $P_1, P_2, P_3, Q_1, Q_2, Q_3$ be six points lying in a conic section; let the areas of the triangles $P_1 Q_2 Q_3, P_1 Q_3 Q_1, P_1 Q_1 Q_2$, be denoted by A_1, B_1, C_1 , and the areas of the triangles formed

by putting P_2, P_3 successively in the place of P_1 be denoted by $A_2, B_2, C_2, A_3, B_3, C_3$ respectively; then will

$$\frac{1}{A_1} \left(\frac{1}{B_2 C_3} - \frac{1}{B_3 C_2} \right) + \frac{1}{A_2} \left(\frac{1}{B_3 C_1} - \frac{1}{B_1 C_3} \right) + \frac{1}{A_3} \left(\frac{1}{B_1 C_2} - \frac{1}{B_2 C_1} \right) = 0.$$

Let u_1, u_2, u_3 denote the distances of any point from the lines $Q_2 Q_3, Q_3 Q_1, Q_1 Q_2$, respectively, then $u_1 = 0, u_2 = 0, u_3 = 0$ will be the equations to these lines themselves. And the equation to any conic section passing through Q_1, Q_2, Q_3 , may be written under the form

$$\frac{\lambda_1}{u_1} + \frac{\lambda_2}{u_2} + \frac{\lambda_3}{u_3} = 0 \dots\dots\dots (1),$$

$\lambda_1, \lambda_2, \lambda_3$ being constants whose values will be determined by the condition of the conic passing through P_1, P_2, P_3 .

Let u'_1, u'_2, u'_3 be the values of u_1, u_2, u_3 respectively at P_1 ,

$$u''_1, u''_2, u''_3 \dots\dots\dots P_2,$$

$$u'''_1, u'''_2, u'''_3 \dots\dots\dots P_3.$$

Then $A_1 = \frac{1}{2} Q_2 Q_3 \cdot u'_1, B_1 = \frac{1}{2} Q_3 Q_1 \cdot u'_2, C_1 = \frac{1}{2} Q_1 Q_2 \cdot u'_3$,

with similar expressions for $A_2, B_2, C_2; A_3, B_3, C_3$:

$$\begin{aligned} \therefore \frac{1}{A_1} \left(\frac{1}{B_2 C_3} - \frac{1}{B_3 C_2} \right) + \frac{1}{A_2} \left(\frac{1}{B_3 C_1} - \frac{1}{B_1 C_3} \right) + \frac{1}{A_3} \left(\frac{1}{B_1 C_2} - \frac{1}{B_2 C_1} \right) \\ = \frac{8}{Q_2 Q_3 \cdot Q_3 Q_1 \cdot Q_1 Q_2} \left\{ \frac{1}{u'_1} \left(\frac{1}{u''_2 u'''_3} - \frac{1}{u'''_2 u''_3} \right) \right. \\ \left. + \frac{1}{u''_1} \left(\frac{1}{u'''_2 u'_3} - \frac{1}{u'''_3 u'_2} \right) + \frac{1}{u'''_1} \left(\frac{1}{u'_2 u''_3} - \frac{1}{u''_2 u'_3} \right) \right\}. \end{aligned}$$

And since $P_1 P_2 P_3$ all lie in (1), we have

$$\frac{\lambda_1}{u'_1} + \frac{\lambda_2}{u'_2} + \frac{\lambda_3}{u'_3} = 0,$$

$$\frac{\lambda_1}{u''_1} + \frac{\lambda_2}{u''_2} + \frac{\lambda_3}{u''_3} = 0,$$

$$\frac{\lambda_1}{u'''_1} + \frac{\lambda_2}{u'''_2} + \frac{\lambda_3}{u'''_3} = 0;$$

whence eliminating $\lambda_1 \lambda_2 \lambda_3$ by cross-multiplication, we get

$$\frac{1}{u_1'} \left(\frac{1}{u_2'' u_3'''} - \frac{1}{u_2''' u_3''} \right) + \frac{1}{u_1''} \left(\frac{1}{u_2''' u_3'} - \frac{1}{u_2' u_3'''} \right) + \frac{1}{u_1'''} \left(\frac{1}{u_2' u_3''} - \frac{1}{u_2'' u_3'} \right) = 0;$$

whence dividing by $\frac{1}{8} Q_2 Q_3 \cdot Q_3 Q_1 \cdot Q_1 Q_2$,

$$\frac{1}{A_1} \left(\frac{1}{B_2 C_3} - \frac{1}{B_3 C_2} \right) + \frac{1}{A_2} \left(\frac{1}{B_3 C_1} - \frac{1}{B_1 C_3} \right) + \frac{1}{A_3} \left(\frac{1}{B_1 C_2} - \frac{1}{B_2 C_1} \right) = 0.$$

5. The equations of three straight lines are

$$u (= x \sin \theta - y \cos \theta + c) = 0, \quad u_1 = 0, \quad u_2 = 0;$$

prove that the equations of the four circles, to each of which these lines are tangents, are

$$u^{\frac{1}{2}} \sin \frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \sin \frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \sin \frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}} \sin \frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \cos \frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \cos \frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}} \cos \frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \sin \frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \cos \frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}} \cos \frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \cos \frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \sin \frac{\theta_1 - \theta}{2} = 0.$$

The equation to any conic section, touched by these three lines, may be written,

$$\lambda u^{\frac{1}{2}} + \lambda_1 u_1^{\frac{1}{2}} + \lambda_2 u_2^{\frac{1}{2}} = 0 \dots\dots\dots (1).$$

If we reduce this equation to the form of an equation of the second degree, the coefficient of the terms of two dimensions will not contain c, c_1, c_2 ; hence, to find the condition of this representing a circle, suppose c, c_1, c_2 to be indefinitely diminished, the ratios $c : c_1 : c_2$ remaining unaltered.

The three lines $u = 0, u_1 = 0, u_2 = 0$, will then all pass through the origin, and the circle touching them will degenerate into the origin; its equation will therefore be

$$x^2 + y^2 = 0,$$

$$\text{or } y = \pm \sqrt{-1} . x.$$

Equation (1) will therefore become, dividing out by x ,

$$\lambda (\sin \theta \mp \frac{1}{2} \cos \theta)^{\frac{1}{2}} + \lambda_1 (\sin \theta_1 \mp \frac{1}{2} \cos \theta_1)^{\frac{1}{2}} + \lambda_2 (\sin \theta_2 \mp \frac{1}{2} \cos \theta_2)^{\frac{1}{2}} = 0 \dots\dots (2)^*.$$

Hence, by Demoivre's theorem, equating real and imaginary parts separately to zero,

$$\left. \begin{aligned} \lambda \sin \frac{\theta}{2} + \lambda_1 \sin \frac{\theta_1}{2} + \lambda_2 \sin \frac{\theta_2}{2} &= 0, \\ \lambda \cos \frac{\theta}{2} + \lambda_1 \cos \frac{\theta_1}{2} + \lambda_2 \cos \frac{\theta_2}{2} &= 0. \end{aligned} \right\} \dots\dots\dots (3).$$

Eliminating λ , λ_1 , λ_2 by cross-multiplication from (1), (3), we get

$$u^{\frac{1}{2}} \sin \frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \sin \frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \sin \frac{\theta_1 - \theta}{2} = 0.$$

Now if in this equation we write $\pi + \theta$ for θ , $\pi + \theta_1$ for θ_1 , and $\pi + \theta_2$ for θ_2 , successively, by which substitution equation (2) is not altered, we get the equations to the remaining circles: these are

$$u^{\frac{1}{2}} \sin \frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \cos \frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \cos \frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}} \cos \frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \sin \frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \cos \frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}} \cos \frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}} \cos \frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}} \sin \frac{\theta_1 - \theta}{2} = 0,$$

the required equations to the circles.

1850.

1. If at a given point two circles intersect and their centres lie upon two lines at right angles to each other through the

* This method of investigating the condition that equation (1) may represent a circle, is due to Mr. Leslie Ellis. It may be shewn in precisely the same manner, that if $\phi(u, u_1, u_2) = 0$ be any equation of the second degree, the condition that this may represent a circle is $\phi(\epsilon^{-\frac{1}{2}\theta}, \epsilon^{-\frac{1}{2}\theta_1}, \epsilon^{-\frac{1}{2}\theta_2}) = 0$.

point; prove that, whatever be the magnitude of the circles, their common tangents will always meet in one of two straight lines which pass through the given point.

Take the given point as origin, and the lines on which the centres lie as axes. Let α, β , be the radii of the circles. Then the intersection of the common tangents must always lie on the line joining the centres of the circles, whose equation is

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1.$$

From considerations of symmetry it is easy to see, that if this intersection always lie on one of two fixed lines passing through the origin, the equations to these lines must be $x + y = 0$, $x - y = 0$. Hence, if such be the case, the equation to one of the common tangents must be

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{x + y}{\gamma} - 1 = 0,$$

where γ is a constant to be determined.

In order that this line may touch the circle whose radius is α , it is necessary and sufficient that its distance from the centre of the circle, whose coordinates are $\alpha, 0$, be α . We must therefore have

$$\frac{\left(\frac{\alpha}{\gamma}\right)^2}{\left(\frac{1}{\alpha} + \frac{1}{\gamma}\right)^2 + \left(\frac{1}{\beta} + \frac{1}{\gamma}\right)^2} = \alpha^2;$$

$$\therefore \frac{1}{\gamma^2} = \left(\frac{1}{\alpha} + \frac{1}{\gamma}\right)^2 + \left(\frac{1}{\beta} + \frac{1}{\gamma}\right)^2,$$

shewing that γ is a symmetrical function of α and β , and therefore that if this straight line touch one of the circles, it must also touch the other. Hence the intersection of the common tangents always lies on the line $x + y = 0$, if α and β have the same sign, *i.e.* if both centres lie on the positive, or both on the negative side of the origin. If one centre lie on the positive, the other on the negative side, similar reasoning will shew that the intersection of the common tangents lies on the line $x - y = 0$.

2. A number n of equal confocal parabolas are ranged all round the focus at equal angular intervals; shew that the product of the distances of all the points of intersection from the focus is $\frac{l^{n(n-1)}}{n^n}$, l being the latus-rectum.

Taking the common focus as pole, the equations to the parabolas will be

$$r = \frac{l}{4 \sin^2 \frac{\theta}{2}} \dots \dots \dots (1),$$

$$r = \frac{l}{4 \sin^2 \left(\frac{\theta}{2} + \frac{\pi}{n} \right)} \dots \dots \dots (2),$$

$$= \dots \dots \dots$$

$$r = \frac{l}{4 \sin^2 \left(\frac{\theta}{2} + \frac{m\pi}{n} \right)} \dots \dots \dots (m+1),$$

$$= \dots \dots \dots$$

$$r = \frac{l}{4 \sin^2 \left(\frac{\theta}{2} + \frac{n-1}{n} \pi \right)} \dots \dots \dots (n).$$

If θ_1, θ_2 be the two values of θ at the intersections of (1) and $(m+1)$ it is manifest that since these intersections lie at the extremities of the same chord passing through the pole,

$$\theta_2 - \theta_1 = \pi.$$

Also we easily see that $\theta_1 = -\frac{m\pi}{n}$,

$$\therefore \theta_2 = \pi - \frac{m\pi}{n}.$$

Hence, if r_1, r_2 be the corresponding values of r ,

$$r_1 = \frac{l}{4 \sin^2 \frac{m\pi}{2n}}, \quad r_2 = \frac{l}{4 \cos^2 \frac{m\pi}{2n}};$$

$$\therefore r_1 r_2 = \frac{l^2}{4 \sin^2 \frac{m\pi}{n}}.$$

Similar expressions holding for the intersection of (1) with each of the other parabolas, we have

product of all the distances of intersections of (1) with the other parabolas

$$= \frac{l^{2(n-1)}}{2^{2(n-1)} \sin^2 \frac{\pi}{n} \sin^2 \frac{2\pi}{n} \dots \sin^2 \frac{n-1}{n} \pi}.$$

$$\text{But} \quad \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \dots \sin \frac{n-1}{n} \pi = \frac{n}{2^{n-1}};$$

$$\text{therefore the above product} = \frac{l^{2(n-1)}}{n^2}.$$

To get the product of all the distances of intersections, we have merely to raise this quantity to the power $\frac{n}{2}$: (not n ; since each intersection would then be counted twice over);

$$\text{therefore product of all the distances of intersections} = \frac{l^{n(n-1)}}{n^n}.$$

3. The locus of the points from which a circle is projected into a circle, upon a plane inclined at a finite angle to that of the given circle, is an equilateral hyperbola.

Let O (fig. 69) be the centre of the given circle, AB that diameter of it in which it is cut by a plane through O , perpendicular to the line of intersection of the plane of the circle, and the plane of projection. Let CD be the corresponding diameter of the circle in which it is projected. Join CA , DB , and produce them to meet in E , E will be the point from which the given circle is projected.

Draw FG parallel to CD and equal to AB , terminated by EC , ED : let FG , AB intersect in P , then must $AP = GP$, $BP = FP$. Through O draw two lines OX , OY , parallel to those respectively bisecting the angles APF , APG , and take them as axes. Let a , b be the coordinates of B ; $-a$, $-b$ of A ; x , y those of E . Then x will be the abscissa of P , and it is hence easy to see that $a + 2x$, $-b$ will be the coordinates of G .

Hence, ξ , η being current coordinates, the equation of ED will be

$$\frac{\xi - a}{a + 2x - a} = \frac{\eta - b}{-b - b},$$

$$\text{or } \frac{\xi - a}{x} = -\frac{\eta - b}{b} \dots\dots\dots (1).$$

And the equation to EP is $\xi = x \dots\dots\dots (2).$

At E , the intersection of these two lines, we have $\eta = y$, and from equations (1), (2),

$$\frac{a}{x} = \frac{y}{b};$$

$$\therefore xy = ab,$$

shewing that the locus of E is an equilateral hyperbola, of which OX , OY are the asymptotes.

4. Prove that $y = nx + \frac{1}{x} + \frac{1}{x + \dots}$ *ad infinitum* is the equation of a hyperbola. Find the position and magnitude of the axes, and write down the equation of the conjugate hyperbola under the same form.

Since $y = nx + \frac{1}{x} + \frac{1}{x + \dots};$

$$\begin{aligned} \therefore y - nx &= \frac{1}{x} + \frac{1}{x + \dots}, \\ &= \frac{1}{x + y - nx}, \\ &= \frac{1}{y - (n-1)x}; \end{aligned}$$

$$\therefore (y - nx) \{y - (n-1)x\} = 1 \dots\dots\dots (1),$$

the equation to an hyperbola, whose asymptotes are represented by the equations

$$y = nx, \quad y = (n-1)x.$$

The axes bisect the angles between the asymptotes, therefore

their equations are

$$\frac{y - nx}{(1 + n^2)^{\frac{1}{2}}} = \frac{y - (n-1)x}{\{1 + (n-1)^2\}^{\frac{1}{2}}} \dots\dots\dots(2),$$

$$\frac{y - nx}{(1 + n^2)^{\frac{1}{2}}} = - \frac{y - (n-1)x}{\{1 + (n-1)^2\}^{\frac{1}{2}}} \dots\dots\dots(3)$$

respectively. Hence the positions of the axes are known.

To find the magnitudes of the axes, we have, combining (1) (2),

$$y - nx = \frac{(1 + n^2)^{\frac{1}{4}}}{\{1 + (n-1)^2\}^{\frac{1}{4}}},$$

$$y - (n-1)x = \frac{\{1 + (n-1)^2\}^{\frac{1}{4}}}{(1 + n^2)^{\frac{1}{4}}};$$

$$\therefore x = \frac{\{1 + (n-1)^2\}^{\frac{1}{2}} - (1 + n^2)^{\frac{1}{2}}}{\{1 + (n-1)^2\}^{\frac{1}{4}}(1 + n^2)^{\frac{1}{4}}},$$

$$\therefore y = \frac{n\{1 + (n-1)^2\}^{\frac{1}{2}} - (n-1)(1 + n^2)^{\frac{1}{2}}}{\{1 + (n-1)^2\}^{\frac{1}{4}}(1 + n^2)^{\frac{1}{4}}},$$

$$\begin{aligned} \therefore x^2 + y^2 &= \frac{2(1 + n^2)\{1 + (n-1)^2\} - 2(n^2 - n + 1)(1 + n^2)^{\frac{1}{2}}\{1 + (n-1)^2\}^{\frac{1}{2}}}{\{1 + (n-1)^2\}^{\frac{1}{2}}(1 + n^2)^{\frac{1}{2}}}, \\ &= 2[(1 + n^2)^{\frac{1}{2}}\{1 + (n-1)^2\}^{\frac{1}{2}} - (n^2 - n + 1)], \end{aligned}$$

which gives the square of the magnitude of one of the semi-axes.

Similarly the square of the other semi-axis may be shewn to be

$$= 2[(1 + n^2)^{\frac{1}{2}}\{1 + (n-1)^2\}^{\frac{1}{2}} + (n^2 - n + 1)].$$

The equation to the conjugate hyperbola is

$$(y - nx)\{y - (n-1)x\} = -1,$$

which may be written

$$y - nx = - \frac{1}{x + y - nx},$$

$$= - \frac{1}{x} - \frac{1}{x - \dots};$$

$$\therefore y = nx - \frac{1}{x} - \frac{1}{x - \dots},$$

is the equation to the conjugate hyperbola.

5. From a point O (fig. 70) are drawn two lines to touch a parabola in the points P and Q ; another line touches the parabola in R and intersects OP , OQ in S , T ; if V be the intersection of the lines joining PT , QS crosswise, O , R , V are in the same straight line.

Let $OP = a$, $OQ = b$, then the equation of the parabola referred to these lines as axes, is

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1 \dots\dots\dots(1).$$

Let $OS = \alpha$, $OT = \beta$, then the equation to ST will be

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1 \dots\dots\dots(2).$$

To find the condition that (2) may touch (1), we proceed as follows:

Square each member of (1) and multiply it crosswise by (2), we thus get

$$\begin{aligned} \frac{x}{\alpha} + \frac{y}{\beta} &= \left\{ \left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} \right\}^2, \\ \text{or } x \left(\frac{1}{\alpha} - \frac{1}{a}\right) + y \left(\frac{1}{\beta} - \frac{1}{b}\right) - 2 \left(\frac{xy}{\alpha\beta}\right)^{\frac{1}{2}} &= 0 \dots\dots(3). \end{aligned}$$

This may be considered as a quadratic in $\left(\frac{y}{x}\right)^{\frac{1}{2}}$, and in order that (2) may touch (1), it is necessary that its roots be equal; hence its first member must be a perfect square.

The equations to PT , QS respectively, are

$$\begin{aligned} \frac{x}{a} + \frac{y}{\beta} &= 1, \\ \frac{x}{\alpha} + \frac{y}{b} &= 1 : \end{aligned}$$

where these meet, we have

$$x \left(\frac{1}{a} - \frac{1}{\alpha}\right) + y \left(\frac{1}{\beta} - \frac{1}{b}\right) = 0,$$

the equation to a line passing through the origin, and through the intersection of PT , QS , that is to OV .

Again, to find the equation to OR , we have since the first member of (3) has been made a perfect square,

$$x^2 \left(\frac{1}{\alpha} - \frac{1}{a} \right)^2 = y^2 \left(\frac{1}{\beta} - \frac{1}{b} \right)^2;$$

therefore squaring, $x \left(\frac{1}{\alpha} - \frac{1}{a} \right) = y \left(\frac{1}{\beta} - \frac{1}{b} \right),$

$$\text{or } x \left(\frac{1}{a} - \frac{1}{\alpha} \right) + y \left(\frac{1}{\beta} - \frac{1}{b} \right) = 0,$$

the equation to OR , which agrees with that already formed for OV ; hence O, R, V are in the same straight line.

6. A series of circles pass through a given point O , have their centres in a line OA , and meet another line AB . From M, N , the points in which one of the circles meets the lines OA, AB , are drawn parallels to AB, OA , intersecting in P . Shew that the locus of P is a hyperbola, which becomes a parabola when the two lines are at right angles.

Take O as origin, OA as axis of x , let the equation to any one of the circles be

$$x^2 + y^2 = 2rx \dots \dots \dots (1),$$

and that to AB $x \cos \alpha + y \sin \alpha = a \dots \dots \dots (2);$

hence the coordinates of M are $2r, 0$, and the equation to the line through M parallel to AB , is

$$x \cos \alpha + y \sin \alpha = 2r \cos \alpha \dots \dots \dots (3).$$

Now the circle in general cuts AB in *two* points, either of which may be denoted by N . The equation to the line through either of these points parallel to OA (the axis of x), will be obtained by putting the ordinate of that point $= 0$, and therefore the equation to the pair of parallels will be obtained by eliminating x between (1) and (2). This gives

$$(a - y \sin \alpha)^2 + y^2 \cos^2 \alpha = 2r \cos \alpha (a - y \sin \alpha);$$

$$\therefore y^2 + 2(r \cos \alpha - a) y \sin \alpha + a^2 - 2ar \cos \alpha = 0 \dots (4).$$

To find the equation to the locus of P , the intersection of (3) with either of these lines, we must eliminate r between (3) and

(4), whence we get

$$y^2 + (x \cos \alpha + y \sin \alpha - 2a) y \sin \alpha + a^2 - a(x \cos \alpha + y \sin \alpha) = 0,$$

or $y^2(1 + \sin^2 \alpha) + xy \cos \alpha \sin \alpha - a(x \cos \alpha + 3y \sin \alpha) + a^2 = 0.$

This is the equation to the locus of P which is evidently in general an hyperbola. If however the lines are at right angles $\alpha = 0$, and the above equation becomes

$$y^2 - ax + a^2 = 0,$$

representing a parabola.

7. If from the focus of a parabola, lines be drawn to meet the tangents at a constant angle, the locus of the points of intersection will be that tangent to the parabola whose inclination to the axis is equal to the given angle. Prove this in any manner, and shew that if m be eliminated between $y = mx + \frac{a}{m}$, and $y = \frac{m+t}{1-mt}(x-a)$, the result contains a factor which answers to the locus. Also explain briefly the origin and signification of the other factors.

(α). Let $4a$ be the latus-rectum of the parabola, α the inclination to the axis of any one of the series of tangents, β the constant angle at which the lines through the focus meet the tangents. Then taking the focus as pole, and the axis as initial line, the equation to the tangent is

$$r = a \operatorname{cosec} \alpha \operatorname{cosec}(\theta + \alpha).$$

That to the line through the focus is

$$\theta = \pi - (\alpha + \beta).$$

Eliminating α , we get as the locus of the intersection of these lines

$$r = a \operatorname{cosec} \beta \operatorname{cosec}(\theta + \beta),$$

representing the tangent whose inclination to the axis = β .

β . From the equation

$$y = \frac{m+t}{1-mt}(x-a) \dots \dots \dots (1),$$

we get

$$m(x-a+ty) = -t(x-a) + y;$$

therefore combining this with

$$y = mx + \frac{a}{m} \dots\dots\dots(2),$$

$$\begin{aligned} y &= \frac{y - t(x-a)}{ty + x - a} x + \frac{ty + x - a}{y - t(x-a)} a, \\ &= \frac{y - t(x-a)}{ty + x - a} (x-a) + a \left\{ \frac{y - t(x-a)}{ty + x - a} + \frac{ty + x - a}{y - t(x-a)} \right\}, \end{aligned}$$

$$\therefore t \frac{y^2 + (x-a)^2}{ty + x - a} = a \frac{\{(x-a)^2 + y^2\} (1+t^2)}{(ty + x - a) \{y - t(x-a)\}};$$

$$\therefore \frac{(x-a)^2 + y^2}{x-a + ty} \cdot \frac{ty - t^2x - a}{y - t(x-a)} = 0.$$

This is satisfied by $y = tx + \frac{a}{t},$

representing the locus found above.

It is also satisfied by

$$(x-a)^2 + y^2 = 0,$$

which requires that $x = a, y = 0$, representing the focus.

This would be obtained by making $m = (-1)^{\frac{1}{2}}$ in equation (2).

Its signification therefore is, that if tangents be drawn to the imaginary branch of the parabola, got by making x negative, and lines be drawn through the focus of the real branch cutting these tangents at a constant angle, the point of intersection of these lines will only be real when the tangent to the imaginary branch of the parabola passes through the focus of the real branch.

8. Within the evolute of an ellipse is inscribed a similar ellipse; within its evolute another similar ellipse, and so on *ad infinitum*; shew that the sum of all the areas

$$= \frac{\pi}{4} \frac{(a^2 + b^2)^2}{ab}.$$

Let ma, mb be the semi-axis of the first inscribed ellipse, then m will be a homogeneous function of a and b of no dimensions, and therefore the same function of ma and mb ; hence

m^2a , m^2b will be those of the second, and the sum of all the areas

$$= \frac{\pi ab}{1 - m^2};$$

we have therefore only to find the value of m .

Now the equation to the evolute of the ellipse is

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}} \dots\dots\dots(1).$$

In order that this may touch the ellipse

$$\left(\frac{x}{ma}\right)^2 + \left(\frac{y}{mb}\right)^2 = 1 \dots\dots\dots(2),$$

they must have a common tangent at a common point.

Now the equation to the tangent to (1) at (xy) is

$$\left(\frac{a^2}{x}\right)^{\frac{1}{3}} x_1 + \left(\frac{b^2}{y}\right)^{\frac{1}{3}} y_1 = (a^2 - b^2)^{\frac{2}{3}}.$$

In order that this may touch (2) at (xy) , we must have

$$\left(\frac{a^2}{x}\right)^{\frac{1}{3}} \frac{1}{(a^2 - b^2)^{\frac{2}{3}}} = \frac{x}{m^2 a^2},$$

$$\left(\frac{b^2}{y}\right)^{\frac{1}{3}} \frac{1}{(a^2 - b^2)^{\frac{2}{3}}} = \frac{y}{m^2 b^2};$$

$$\therefore \frac{x^2}{m^2 a^2} = \frac{ma^2}{a^2 - b^2},$$

$$\frac{y^2}{m^2 b^2} = \frac{mb^2}{a^2 - b^2};$$

$$\text{therefore, by (2), } m = \frac{a^2 - b^2}{a^2 + b^2};$$

therefore if S equals the sum of all the areas,

$$\begin{aligned} S &= \frac{\pi ab}{1 - \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2} \\ &= \frac{\pi}{4} \frac{(a^2 + b^2)^2}{ab}. \end{aligned}$$

9. Find the points $A_1, A_2, A_3 \dots A_{m-1}, A_m$ in a parabola, such that the tangents at these points are parallel to the focal distances $SA_m, SA_1, SA_2, \dots SA_{m-2}, SA_{m-1}$, respectively.

Let $A_m A_1 K V$ (fig. 71) represent the parabola, $K S X$ its axis, $A_1 T_1, \dots A_m T_m$ the tangents at $A_1, \dots A_m$ respectively, and let $\angle A_r S X = \alpha_r$, then $\angle A_r T_r X = \frac{1}{2} \alpha_r$. Hence, by the conditions of the problem, we must have

$$\frac{1}{2} \alpha_1 = \alpha_m \dots\dots\dots (1),$$

$$\frac{1}{2} \alpha_2 = \alpha_1 \dots\dots\dots (2),$$

$$\dots = \dots\dots\dots$$

$$\frac{1}{2} \alpha_r = \alpha_{r-1} \dots\dots\dots (r),$$

$$\dots = \dots\dots\dots$$

$$\frac{1}{2} \alpha_{m-1} = \alpha_{m-2} \dots\dots\dots (m-1).$$

The last equation will be

$$\pi + \frac{1}{2} \alpha_m = \alpha_{m-1} \dots\dots\dots (m),$$

since this will satisfy the condition of $A_m T_m$ being parallel to SA_{m-1} , and it is manifestly inconsistent with the preceding equations to have $\frac{1}{2} \alpha_m = \alpha_{m-1}$.

Hence, multiplying generally equation (r) by 2^{m-r} , and adding all equations thus formed, the quantities $\alpha_1, \alpha_2, \dots \alpha_{m-1}$, disappear, and we get

$$\pi + \frac{1}{2} \alpha_m = 2^{m-1} \alpha_m;$$

$$\therefore \alpha_m = \frac{2\pi}{2^m - 1};$$

$$\text{whence } \alpha_1 = \frac{2^2 \pi}{2^m - 1},$$

$$\text{and generally, } \alpha_r = \frac{2^{r+1} \pi}{2^m - 1},$$

and the positions of the points $A_1, A_2 \dots A_m$ are determined.

10. From the focus of an ellipse lines are drawn to any four points in the curve, and the reciprocal of each line is multiplied by the sines of half the angles between any two of the remaining lines; prove that the sum of the first and third of these products taken in order is equal to the sum of the second and fourth.

Let L = the latus-rectum, e the eccentricity of the ellipse; then its equation, referred to the focus as origin and the axis-major as prime radius, will be

$$\frac{1}{r} = \frac{1 - e \cos \theta}{\frac{1}{2}L};$$

and let $\theta_1, \theta_2, \theta_3, \theta_4$ be the angles between the axis and the successive lines, r_1, r_2, r_3, r_4 the lengths of the lines.

$$\begin{aligned} \text{Then} \quad & \frac{1}{r_1} \sin \frac{\theta_3 - \theta_4}{2} \sin \frac{\theta_4 - \theta_2}{2} \sin \frac{\theta_2 - \theta_3}{2} \\ &= \frac{2}{L} (1 - e \cos \theta_1) \sin \frac{\theta_3 - \theta_4}{2} \sin \frac{\theta_4 - \theta_2}{2} \sin \frac{\theta_2 - \theta_3}{2} \\ &= \frac{1}{2L} (1 - e \cos \theta_1) \{ \sin (\theta_4 - \theta_3) + \sin (\theta_2 - \theta_4) + \sin (\theta_3 - \theta_2) \}. \\ \text{Similarly} \quad & \frac{1}{r_3} \sin \frac{\theta_1 - \theta_2}{2} \sin \frac{\theta_2 - \theta_4}{2} \sin \frac{\theta_4 - \theta_1}{2} \\ &= \frac{1}{2L} (1 - e \cos \theta_3) \{ \sin (\theta_2 - \theta_1) + \sin (\theta_4 - \theta_2) + \sin (\theta_1 - \theta_4) \}; \end{aligned}$$

therefore if S denote the sum of these two quantities, and if the symmetrical function $\sin (\theta_4 - \theta_3) + \sin (\theta_2 - \theta_4) + \sin (\theta_3 - \theta_2) + \sin (\theta_2 - \theta_1) + \sin (\theta_4 - \theta_2) + \sin (\theta_1 - \theta_4)$ be denoted by ϕ , and if we put c_1 for $\cos \theta_1$, s_1 for $\sin \theta_1$, &c.,

$$\begin{aligned} S &= \frac{1}{2L} [\phi - e \{ c_1 (s_4 c_3 - s_3 c_4 + s_2 c_4 - s_4 c_2 + s_3 c_2 - s_2 c_3) \\ &\quad + c_3 (s_2 c_1 - s_1 c_2 + s_4 c_2 - s_2 c_4 + s_1 c_4 - s_4 c_1) \}] \\ &= \frac{1}{2L} [\phi - e \{ c_1 c_2 (s_3 - s_4) + c_2 c_3 (s_4 - s_1) + c_3 c_4 (s_1 - s_2) + c_4 c_1 (s_2 - s_3) \}]. \end{aligned}$$

Similarly, if S' be the sum of the second and fourth products, S' will be obtained from S by writing θ_2 for θ_1 , θ_3 for θ_2 , θ_4 for θ_3 , θ_1 for θ_4 ; hence ϕ , which is a symmetrical function, will remain unchanged, and

$$S' = \frac{1}{2L} [\phi - e \{ c_2 c_3 (s_4 - s_1) + c_3 c_4 (s_1 - s_2) + c_4 c_1 (s_2 - s_3) + c_1 c_2 (s_3 - s_4) \}];$$

whence it appears that $S = S'$,

or the sum of the first and third of these products is equal to the sum of the second and fourth.

11. If lines be drawn through any two of the points A, B, C, \dots and other lines through any two of the points a, b, c, \dots all in one plane, prove that the intersections of AB with ab , of AC with ac, \dots , will all lie in one straight line, provided that the lines through the intersections of any two of the first series of lines and the corresponding intersections of the second series all pass through the same point.

Conceive the points A, B, C, \dots to lie in one plane, and a, b, c, \dots in another; then, since the lines joining the intersections of any two of the first series of lines and the corresponding intersections of the second series all pass through one point, Aa, Bb, Cc, \dots all pass through one point O .

Now consider any quadrilateral, as $ABab$, whose angular points are any two points of the first series, and the corresponding two of the second series. Since the lines Aa, Bb intersect, they are in the same plane, therefore also AB, ab , are in the same plane, and must therefore intersect,* and their intersection must manifestly lie in the line of intersection of the planes ABC, \dots, abc, \dots . Similarly the intersection of any other pair of lines, as AC, ac , lies in that line.

Hence, if we suppose the planes ABC, \dots, abc, \dots , to be indefinitely nearly coincident, the proposition enunciated follows at once.

1851.

1. Having given a focus and two tangents of a conic section, shew by means of reciprocal polars, or otherwise, that the chord of contact always passes through a fixed point.

Let a circle be described passing through two fixed points, A, B , and let P be the intersection of the tangents at A, B . The locus of P will be a fixed straight line, perpendicular to and bisecting AB .

Now take the polar reciprocal of this system with respect to any fixed point S . The reciprocal of the circle will be a conic section whose focus is S , and which has two fixed tangents (the reciprocals of A, B). Hence the reciprocal of P , which is

* If these lines happen to be parallel we may still consider them as intersecting in a point infinitely distant.

the chord of contact of these tangents, will always pass through a fixed point, the reciprocal of the locus of P .

2. Shew that there will be two pairs of equilateral hyperbolæ which pass through two given points A, B , and touch two given straight lines, and that the chords of contact of each pair meet in AB , and are equally inclined to AB .*

Take the middle point of AB as origin, AB as axis of x , let $h, -h$, be the abscissæ of A, B , respectively, and let the equations to the two given tangents be

$$\frac{x}{a} + \frac{y}{b} - 1 = 0, \quad \frac{x}{a'} + \frac{y}{b'} - 1 = 0,$$

and that to their chord of contact $\frac{x}{\alpha} + \frac{y}{\beta} - 1 = 0$,

where α, β , are indeterminate parameters.

Then the equation to a conic section touching the two given lines may be written under the form

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right) \left(\frac{x}{a'} + \frac{y}{b'} - 1\right) = \lambda \left(\frac{x}{\alpha} + \frac{y}{\beta} - 1\right)^2,$$

λ being an indeterminate parameter.

Two equations for the determination of the three arbitrary quantities λ, α, β , are given by the conditions of its passing through A, B . We thus get

$$\left(\frac{h}{a} - 1\right) \left(\frac{h}{a'} - 1\right) = \lambda \left(\frac{h}{\alpha} - 1\right)^2 \dots\dots\dots(1),$$

$$\left(\frac{h}{a} + 1\right) \left(\frac{h}{a'} + 1\right) = \lambda \left(\frac{h}{\alpha} + 1\right)^2 \dots\dots\dots(2).$$

The third equation is given by the condition of the curve being an equilateral hyperbola. In order that this may be the case, it is necessary that the sum of the coefficients of x^2 and $y^2 = 0$. This gives

$$\frac{1}{aa'} - \frac{\lambda}{\alpha^2} + \frac{1}{bb'} - \frac{\lambda}{\beta^2} = 0 \dots\dots\dots(3).$$

* A shorter solution of this problem, due to Mr. Gaskin, will be found in the Appendix.

Combining (1) and (2), we get

$$\left(\frac{h-\alpha}{h+\alpha}\right)^2 = \frac{(h-a)(h-a')}{(h+a)(h+a')}.$$

This is a quadratic for the determination of α . Let α_1, α_2 be its roots.

Subtracting (1) from (2) and eliminating λ by (3), we get

$$\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) \left(\frac{1}{a} + \frac{1}{a'}\right) = \frac{2}{\alpha} \left(\frac{1}{aa'} + \frac{1}{bb'}\right).$$

Hence, for each value of α there will be two of β , equal and of opposite signs. Let them be denoted by $\beta_1, -\beta_1, \beta_2, -\beta_2$ respectively, then we get two pairs of equilateral hyperbolæ, whose chords of contact respectively are

$$\frac{x}{\alpha_1} + \frac{y}{\beta_1} = 1, \quad \frac{x}{\alpha_1} - \frac{y}{\beta_1} = 1 \dots\dots\dots (4),$$

$$\frac{x}{\alpha_2} + \frac{y}{\beta_2} = 1, \quad \frac{x}{\alpha_2} - \frac{y}{\beta_2} = 1 \dots\dots\dots (5).$$

It is easy to see that the lines represented by (4) intersect in AB (the axis of x) and are equally inclined to AB . The same will be the case with the pair of lines denoted by (5). Hence the chords of contact of each pair meet in AB , and are equally inclined to AB .

3. If from a point of an ellipse a line be drawn to the extremity of each axis, and a parallel to the same axis be drawn through the point in which such line meets the other axis, the locus of the intersection of these parallels is an equilateral hyperbola.

Trace the corresponding positions of the point on the hyperbola, and the point on the ellipse.

Let a, b , be the semi-axes of the ellipse, take the axes of the curve as coordinate axes, and let $a \cos \alpha, b \sin \alpha$, (α being a variable parameter) be the coordinates of the point through which the lines are drawn. The equation to the line drawn through this point to the extremity of the axis-major, is

$$\frac{x - a \cos \alpha}{a \cos \alpha - a} = \frac{y - b \sin \alpha}{b \sin \alpha},$$

$$\text{or } \frac{x - a \cos \alpha}{a \sin \frac{1}{2} \alpha} + \frac{y - b \sin \alpha}{b \cos \frac{1}{2} \alpha} = 0;$$

$$\therefore \frac{x}{a} \cos \frac{\alpha}{2} + \frac{y}{b} \sin \frac{\alpha}{2} = \cos \frac{\alpha}{2}.$$

Where this meets the axis-minor,

$$y = b \cot \frac{1}{2} \alpha \dots\dots\dots (1),$$

which is therefore the equation to the line through this point, parallel to the axis-major.

Again, the equation to the line drawn through $(a \cos \alpha, b \sin \alpha)$ to the extremity of the axis-minor, is

$$\frac{y - b \sin \alpha}{b \sin \alpha - b} = \frac{x - a \cos \alpha}{a \cos \alpha},$$

$$\text{or } \frac{y - b \cos(\frac{1}{2} \pi - \alpha)}{b \cos(\frac{1}{2} \pi - \alpha) - b} = \frac{x - a \sin(\frac{1}{2} \pi - \alpha)}{a \sin(\frac{1}{2} \pi - \alpha)};$$

whence, by an investigation similar to the above, it is seen that the equation to the line drawn parallel to the axis-minor through the point where this meets the axis-major, is

$$\begin{aligned} x &= a \cot(\frac{1}{4} \pi - \frac{1}{2} \alpha) \\ &= a \frac{\cot \frac{1}{2} \alpha + 1}{\cot \frac{1}{2} \alpha - 1} \dots\dots\dots (2). \end{aligned}$$

Eliminating α between (1) and (2), we get

$$x = a \frac{y + b}{y - b},$$

$$\text{or } xy = ay + bx + ab,$$

$$\text{or } (x - a)(y - b) = 2ab,$$

as the equation to the locus, which is evidently an equilateral hyperbola, whose asymptotes are the tangents to the ellipse at A and B .

Let $ABA'B'$ (fig. 72) be the ellipse, P the point $(a \cos \alpha, b \sin \alpha)$; then from the figure it appears that p is the point on the hyperbola corresponding to P : one branch of the hyperbola is described, while P moves from A to B ; the other branch while p moves round through $BA'B'A$.

4. Determine the values of α' , m' , n' , such that the relation

$$\{(x - \alpha')^2 + y^2\}^{\frac{1}{2}} = m' (x^2 + y^2)^{\frac{1}{2}} + n'$$

may be equivalent to the relation

$$\{(x - \alpha)^2 + y^2\}^{\frac{1}{2}} = m (x^2 + y^2)^{\frac{1}{2}} + n.$$

The transformation fails (1) in the case where the curve represented by the given equation is a conic section, (2) has a double point.

(α). The equation

$$\{(x - \alpha')^2 + y^2\}^{\frac{1}{2}} = m' (x^2 + y^2)^{\frac{1}{2}} + n',$$

when transformed to polar coordinates, and rationalized, becomes

$$r^2 - 2\alpha' r \cos \theta + \alpha'^2 = (m' r + n')^2,$$

$$\text{or } (1 - m'^2) r^2 - 2\alpha' r \cos \theta - 2m' n' r + \alpha'^2 - n'^2 = 0.$$

The equation

$$\{(x - \alpha)^2 + y^2\}^{\frac{1}{2}} = m (x^2 + y^2)^{\frac{1}{2}} + n,$$

similarly transformed, becomes

$$(1 - m^2) r^2 - 2\alpha r \cos \theta - 2mn r + \alpha^2 - n^2 = 0.$$

In order that these equations may be identical, the coefficients of r^2 , $r \cos \theta$, r , must bear the same ratio to one another as the constant terms,

$$\therefore \frac{1 - m'^2}{1 - m^2} = \frac{\alpha'}{\alpha} = \frac{m' n'}{mn} = \frac{\alpha'^2 - n'^2}{\alpha^2 - n^2}.$$

From the equation
$$\frac{\alpha'}{\alpha} = \frac{\alpha'^2 - n'^2}{\alpha^2 - n^2}$$

$$= \frac{\alpha'^2}{\alpha^2} \frac{1 - \frac{n'^2}{\alpha'^2}}{1 - \frac{n^2}{\alpha^2}},$$

we get
$$\frac{\alpha'}{\alpha} = \frac{1 - \frac{n^2}{\alpha^2}}{1 - \frac{n'^2}{\alpha'^2}};$$

$$\therefore \frac{1 - \frac{n^2}{\alpha^2}}{1 - \frac{n'^2}{\alpha'^2}} = \frac{1 - m'^2}{1 - m^2}.$$

Also we have $\frac{\alpha'}{\alpha} = \frac{m'n'}{mn}$, or $\frac{m'}{m} = \frac{\frac{n}{\alpha'}}{\frac{n}{\alpha}}$;

$$\therefore m' = \frac{n}{\alpha} \dots\dots\dots (1),$$

$$\text{and } \frac{n'}{\alpha'} = m,$$

$$\text{whence } \frac{\alpha'}{\alpha} = \frac{1 - \frac{n^2}{\alpha'^2}}{1 - m^2};$$

$$\therefore \alpha' = \frac{1}{\alpha} \frac{\alpha^2 - n^2}{1 - m^2} \dots\dots\dots (2),$$

$$n' = \frac{m}{\alpha} \frac{\alpha^2 - n^2}{1 - m^2} \dots\dots\dots (3).$$

The required values of α' , m' , n' , are determined from (1), (2), and (3).

(β). The given equation, when rationalized, will in general be of the fourth degree in x and y . In order therefore that it may represent a conic section, it is necessary that the terms of a degree higher than the second should disappear of themselves from the rationalized equation, which requires that $m = \pm 1$. If this condition be satisfied, the given equation becomes

$$\{(x - \alpha)^2 + y^2\}^{\frac{1}{2}} \pm (x^2 + y^2)^{\frac{1}{2}} = n,$$

shewing that one of the foci of the conic section is the origin, that the coordinates of the other are α , 0, and that the axis-major = n .

The transformed equation must therefore, since it represents a conic section, take the form

$$\{(x - \alpha')^2 + y^2\}^{\frac{1}{2}} \pm (x^2 + y^2)^{\frac{1}{2}} = n',$$

which gives α' , 0 as the coordinates of the second focus, n' as the axis-major. Hence we must have

$$\alpha' = \alpha, \quad n' = n,$$

and the transformation fails.

The transformation also fails if $n = \alpha$, for we then get $m' = 1$, $\alpha' = 0$, $n' = 0$, and the transformed equation becomes an identity. In this case, the given equation, transformed into polar coordinates, becomes

$$(1 - m^2) r - 2(\alpha \cos \theta + m\alpha) = 0;$$

whence we see that if

$$\cos \theta = -m,$$

we have $r = 0$; but the equation $\cos \theta = -m$ is satisfied in general by two values of θ , whose sum $= 2\pi$; hence the origin is a double point.

If therefore the curve have a double point, the transformation fails again.

5. If O be the centre of a reflecting circle, Q a radiant point, and the line from Q to O produced to meet the circle be considered as the axis, then, if a be the radius, u the distance QO , θ the inclination to the axis of the radius through the point of incidence of any ray, and ϕ the inclination to the axis of the reflected ray,

$$\rho \cos \phi = a \cos \theta + u \cos 2\theta, \quad \rho \sin \phi = a \sin \theta + u \sin 2\theta,$$

where $\rho = (a^2 + u^2 + 2au \cos \theta)^{\frac{1}{2}}$ is the length of the incident ray.

Let P (fig. 73) be the point of incidence of any ray, M the point in which the reflected ray cuts the axis. Let

$$\angle QPO = \angle OPM = \psi,$$

and draw PN perpendicular to the axis. Then

$$QN = \rho \cos(\phi - 2\psi) = QO + ON = u + a \cos \theta,$$

$$PN = \rho \sin(\phi - 2\psi) = a \sin \theta.$$

$$\text{Again, } \psi = \phi - \theta,$$

$$\therefore \rho \cos(2\theta - \phi) = u + a \cos \theta \dots\dots\dots (1),$$

$$\rho \sin(2\theta - \phi) = a \sin \theta \dots\dots\dots (2):$$

(1) $\cos 2\theta$ + (2) $\sin 2\theta$ gives

$$\rho \cos \phi = a \cos \theta + u \cos 2\theta,$$

(1) $\sin 2\theta - (2) \cos 2\theta$ gives

$$\rho \sin \phi = a \sin \theta + u \sin 2\theta,$$

the required equations.

6. Using the notation of the last question, and assuming the truth of the theorem stated therein, shew that if from the point of incidence of each ray there be drawn, in a direction opposite to that of the reflected ray, a line equal in length to the incident ray, the locus of the extremities of these lines is a curve cutting the lines at right angles, and the equation of which, referred to the radiant point as origin and the axis QO as axis of x , is

$$x^2 + y^2 - 2ux = 2a(x^2 + y^2)^{\frac{1}{2}}.$$

Shew that the origin is a double point, and trace the curve: shew also that the equation may be expressed in the form

$$\{(x-a)^2 + y^2\}^{\frac{1}{2}} = m(x^2 + y^2)^{\frac{1}{2}} + n.$$

(α). Produce MP to R , making $PR = QP$, then we have to find the locus of R . Let x, y be its coordinates, then we readily see that

$$\begin{aligned} x &= QN + \rho \cos \phi \\ &= u + a \cos \theta + a \cos \theta + u \cos 2\theta \\ &= u(1 + \cos 2\theta) + 2a \cos \theta, \end{aligned}$$

$$\begin{aligned} \text{and } y &= PN + \rho \sin \phi \\ &= a \sin \theta + a \sin \theta + u \sin 2\theta \\ &= u \sin 2\theta + 2a \sin \theta: \end{aligned}$$

hence $\frac{y}{x} = \tan \theta$, and RQ is parallel to PO ,

$$\begin{aligned} \text{and } (x-u)^2 + y^2 &= u^2 + 4a^2 + 4au \cos \theta, \\ &= u^2 + 4a^2 + 4au \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}; \end{aligned}$$

$$\therefore x^2 + y^2 - 2ux = 4a^2 + 4au \frac{x}{(x^2 + y^2)^{\frac{1}{2}}},$$

$$\therefore x^2 + y^2 - 2ux + u^2 \frac{x^2}{x^2 + y^2} = 4a^2 + 4au \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} + u^2 \frac{x^2}{x^2 + y^2},$$

$$\therefore (x^2 + y^2)^{\frac{1}{2}} - \frac{ux}{(x^2 + y^2)^{\frac{1}{2}}} = 2a + \frac{ux}{(x^2 + y^2)^{\frac{1}{2}}},$$

$$\therefore x^2 + y^2 - 2ux = 2a(x^2 + y^2)^{\frac{1}{2}},$$

the required equation to the curve.

(β). This equation, transformed to polar coordinates, becomes

$$r - 2u \cos \theta = 2a,$$

$$\therefore r = 2u \cos \theta + 2a \dots \dots \dots (1).$$

Hence the radius vector of the curve exceeds by $2a$ the radius vector of a circle which passes through the pole, one of whose diameters is prime radius, and whose radius $= u$. If therefore we draw such a circle, and produce OA^* , the radius vector of any point A in it to R , making $AR = 2a$, the locus of R will be the required curve.

The curve will pass through the origin when $\cos \theta = -\frac{a}{u}$, which condition, if $a < u$, is satisfied by two values of θ , one less, the other greater than π . Hence if $a < u$, *i.e.* if Q be outside the circle, two branches of the curve pass through the origin, which is therefore a double point.

When $\theta = 0$, $r = 2(u + a)$, and when $\theta = \pi$, $r = -2(u - a)$; hence the curve will have the form represented in fig. (74), where

$$Qq = 2(u + a), \quad Qq' = 2(u - a). \dagger$$

Again, $\angle QRM = \angle OPM$ (since QR is parallel to OP) $\psi = \phi - \theta$,

$$\begin{aligned} \therefore \tan QRM &= \frac{\sin \phi \cos \theta - \cos \phi \sin \theta}{\cos \phi \cos \theta + \sin \phi \sin \theta} \\ &= -\frac{u \sin \theta}{a + u \cos \theta}, \end{aligned}$$

by the result of question 5.

* The line OA must always be produced in the *positive* direction of the radius vector, therefore when $\theta > \frac{1}{2}\pi$, OA must be produced backwards.

† This is the form of the figure when Q is outside the circle: if it be within it, the curve does not pass through the origin, and the loop Qq' does not appear. The origin will then be a conjugate point.

And if ι be the angle between the radius vector and tangent,

$$\cot \iota = \frac{1}{\iota} \frac{du}{d\theta} = - \frac{u \sin \theta}{a + u \cos \theta},$$

$$= \tan QRM.$$

Hence the curve cuts the lines PR at right angles.

(γ). The equation

$$\{(x - \alpha)^2 + y^2\}^{\frac{1}{2}} = m(x^2 + y^2)^{\frac{1}{2}} + n$$

is equivalent to

$$(1 - m^2)r^2 - 2mnr - 2\alpha r \cos \theta + \alpha^2 - n^2 = 0,$$

and this coincides with (1), if

$$\frac{mn}{1 - m^2} = \alpha, \quad \frac{\alpha}{1 - m^2} = u, \quad \alpha^2 - n^2 = 0;$$

$$\therefore m = \frac{\alpha}{u}, \quad n = \alpha = \frac{u^2 - \alpha^2}{u},$$

therefore (1) is equivalent to

$$\left\{ \left(x - \frac{u^2 - \alpha^2}{u} \right)^2 + y^2 \right\}^{\frac{1}{2}} = \frac{u}{\alpha} (x^2 + y^2)^{\frac{1}{2}} + \frac{u^2 - \alpha^2}{u},$$

which is in the required form.

7. Given the centres of three circles, each of them touching the other two externally, determine the radii.

How many systems of circles are there when the centres are given, but the circles touch externally or internally at pleasure?

Let a, b, c , be the distances between the given centres; then

$$r_2 + r_3 = a,$$

$$r_3 + r_1 = b,$$

$$r_1 + r_2 = c,$$

$$\therefore r_1 = \frac{b + c - a}{2};$$

$$\text{similarly } r_2 = \frac{c + a - b}{2},$$

$$r_3 = \frac{a + b - c}{2},$$

which determine the radii.

If the circles touch internally or externally at pleasure, there will be four systems. For the circle described with any one point as centre may include the other two, which must touch each other externally, thus giving three systems. Or they may all touch externally, giving four systems in all.

8. The locus of the point from which two given circles subtend equal angles is a circle.

Let A, A' (fig. 75) be the centres of the two given circles, P a point from which the circles subtend equal angles. Draw the tangents PT, Pt to the circle whose centre is A ; PT', Pt' to that whose centre is A' . Join $PA, PA', AT, At, A'T', A't'$. Then $\angle TPt = \angle T'Pt'$.

And $AP, A'P$ respectively bisect the angles $TPt, T'Pt'$,

$$\therefore \angle APT = \angle A'PT':$$

also the right angle $ATP =$ the right angle $A'T'P$; therefore the triangles $TAP, T'A'P$ are similar, therefore

$$AP : A'P :: AT : A'T' \dots\dots\dots (1).$$

Divide AA' in O , so that $AO : A'O :: AT : A'T'$ take O as origin, AOA' as axis of x . Let $AO = a, A'O = a'$, and let x, y be the coordinates of P . Then by (1)

$$\{(x+a)^2 + y^2\}^{\frac{1}{2}} : \{(x-a')^2 + y^2\}^{\frac{1}{2}} :: a : a';$$

$$\therefore a^2 \{(x+a)^2 + y^2\} = a'^2 \{(x-a')^2 + y^2\},$$

$$\text{or } (a-a')(x^2 + y^2) - 2aa'x = 0;$$

shewing that the locus of P is a circle passing through O .

9. The lines joining the corresponding points of two similar and similarly situated figures in the same plane intersect in a point.

All sections of a conical surface of any degree by parallel planes are similar and similarly situated figures, and every generating line passes through corresponding points. Hence, conversely, the lines joining corresponding points of two similar and similarly situated figures in parallel planes, pass through one point (the vertex of the conical surface of which they are

sections). Let the planes be now made indefinitely nearly coincident, and the proposition enunciated follows at once.

10. Given any three of the four lines Ox , Oy , Op , Oq , (fig. 76), the fourth may be determined, such that if a , b be the points in which a line through a point Q in Oq intersect Ox , Oy , and a' , b' the points in which another line through the same point Q intersects Ox , Oy , the point of intersection of the lines ab' and $a'b$ lies on the line Op .

Let $u = 0$, $v = 0$, be the equations to any two lines passing through the point O , and let $u = \lambda_x v$, $u = \lambda_y v$, $u = \lambda_p v$, $u = \lambda_q v$, be the equations to Ox , Oy , Op , Oq , respectively. Also let $w = 0$ be the equation to Qa , and $u - \lambda_q v - \mu w = 0$ that to Qa' .

Then the equation

$$\alpha(u - \lambda_x v) - (u - \lambda_q v - \mu w) = 0,$$

where α is a disposable parameter, represents a line passing through a' . In order that this may pass through b , the above equation must be identical with

$$\beta(u - \lambda_y v) - w = 0,$$

β being also a disposable quantity. In order that these equations may be identical, we must have

$$\frac{\alpha - 1}{\beta} = \frac{\alpha \lambda_x - \lambda_q}{\beta \lambda_y} = -\mu;$$

$$\therefore \alpha = \frac{\lambda_y - \lambda_q}{\lambda_y - \lambda_x},$$

$$\text{and } (\lambda_y - \lambda_q)(u - \lambda_x v) - (\lambda_y - \lambda_x)(u - \lambda_y v - \mu w) = 0,$$

$$\text{or } (\lambda_x - \lambda_q)(u - \lambda_y v) + (\lambda_y - \lambda_x)\mu w = 0,$$

is the equation to $a'b$. Similarly

$$(\lambda_y - \lambda_q)(u - \lambda_x v) + (\lambda_x - \lambda_y)\mu w = 0,$$

is that to ab' . Where these intersect, we have

$$(\lambda_x - \lambda_q)(u - \lambda_y v) + (\lambda_y - \lambda_q)(u - \lambda_x v) = 0,$$

$$\text{or } (\lambda_x + \lambda_y - 2\lambda_q)u + \lambda_q(\lambda_x + \lambda_y)v = 0 \dots \dots \dots (1).$$

In order that this may lie in the line Op , whose equation is

$$u - \lambda_p v = 0 \dots \dots \dots (2),$$

(1) and (2) must be identical; hence

$$\lambda_p(\lambda_x + \lambda_y - 2\lambda_q) + \lambda_q(\lambda_x + \lambda_y) = 0,$$

$$\text{or } (\lambda_p + \lambda_q)(\lambda_x + \lambda_y) - 2\lambda_p\lambda_q = 0,$$

an equation from which, when any three of the quantities $\lambda_x, \lambda_y, \lambda_p, \lambda_q$, are given, the fourth may be determined, so that when any three of the lines Ox, Oy, Op, Oq are given, the fourth may be determined so as to satisfy the required conditions.

11. The radii vectores from the focus of a conic section to two points of the curve make equal angles with the line drawn from the focus to the point of intersection of the tangents at the two points.

Let α, β be the angles which the radii vectores respectively make with the axis-major, then the polar equations to the tangents, referred to the focus as pole and the axis-major as prime radius, will be

$$\frac{1}{r} = \frac{2}{l} \{e \cos \theta + \cos(\theta - \alpha)\},$$

$$\frac{1}{r} = \frac{2}{l} \{e \cos \theta + \cos(\theta - \beta)\}.$$

Where these meet, we must have

$$\theta - \alpha = -(\theta - \beta),$$

$$\therefore \theta = \frac{\alpha + \beta}{2};$$

which will be the equation to the line through the focus and the intersection of the tangents, which evidently bisects the angle between the radii vectores. Hence the proposition is true.

For a demonstration of this theorem by the method of reciprocal polars, see Salmon's *Conic Sections*, chap. XIV.

12. If two triangles be circumscribed about a conic section, their angular points lie in another conic section.

Let $u = 0, v = 0, w = 0$, be the equations to the sides of one triangle, and let the sides of the other triangle respectively opposite to these be represented by

$$u + b_1v + c_1w = 0 \dots (1), \quad a_2u + v + c_2w = 0 \dots (2), \quad a_3u + b_3v + w = 0 \dots (3).$$

Then, since these two triangles are circumscribed about a conic section, it will follow that if (v_3w_2) denote the line joining the intersection of $v = 0$ and (3) with that of $w = 0$ and (2), with similar notation for the other corresponding lines, (v_3w_2) , (w_1u_3) , (u_2v_1) all pass through one point.

Now the equation to (v_3w_2) is

$$\left. \begin{array}{l} u + \frac{v}{a_2} + \frac{w}{a_3} = 0 \\ \text{that to } w_1u_3 \text{ is } \frac{u}{b_1} + v + \frac{w}{b_3} = 0 \\ \text{to } u_2v_1, \frac{u}{c_1} + \frac{v}{c_2} + w = 0 \end{array} \right\} \dots\dots\dots (A).$$

The elimination of u, v, w between these equations would give the necessary condition that the three lines denoted by them should pass through one point, or that the two triangles should be circumscribed about a conic section.

Now the equation to any conic circumscribing the triangle (123) can be put into the form

$$\alpha(a_2u + v + c_2w)(a_3u + b_3v + w) + \beta(a_3u + b_3v + w)(u + b_1v + c_1w) \\ + \gamma(u + b_1v + c_1w)(a_2u + v + c_2w) = 0 \dots\dots\dots (4).$$

Here the coefficient of u^2 is proportional to

$$\alpha + \frac{\beta}{a_2} + \frac{\gamma}{a_3},$$

$$\text{of } v^2 \text{ to } \frac{\alpha}{b_1} + \beta + \frac{\gamma}{b_3},$$

$$\text{of } w^2 \text{ to } \frac{\alpha}{c_1} + \frac{\beta}{c_2} + \gamma.$$

Hence, if we give to α, β, γ respectively the values which u, v, w have at the intersection of the lines (A), each of these coefficients will vanish, and equation (4) will be reduced to one involving vw, wu, uv only; it will therefore also represent a conic circumscribing the triangle whose sides are $u = 0, v = 0, w = 0$; and consequently, if two triangles be circum-

scribed about a conic section, their angular points lie in another conic section.*

From the above proof it is not difficult to see that the converse (which is also the reciprocal theorem) is true.

13. If the angles ϕ, ϕ' are connected by the equation

$$\cos \mu = \cos \phi \cos \phi' - \sin \phi \sin \phi' (1 - c^2 \sin^2 \mu)^{\frac{1}{2}},$$

and $\sin \phi, \sin \phi'$ are the abscissæ of points on an ellipse, the semi-axes of which are 1, $(1 - c^2)^{\frac{1}{2}}$, then the tangents at these points meet in a point, the locus of which is an ellipse confocal with the given ellipse.

Let ξ, η be the coordinates of the intersection of the tangents, then the equation to its polar is

$$\xi x + \frac{\eta y}{1 - c^2} = 1.$$

Let $\sin \theta, (1 - c^2)^{\frac{1}{2}} \cos \theta$, be the coordinates of the points where this line meets the given ellipse, then

$$\xi \sin \theta + \frac{\eta \cos \theta}{(1 - c^2)^{\frac{1}{2}}} = 1;$$

the roots of this equation in θ are ϕ, ϕ' .

It may be written in the form

$$(\xi \sin \theta - 1)^2 = \frac{\eta^2}{(1 - c^2)} (1 - \sin^2 \theta);$$

$$\therefore \sin \phi \cdot \sin \phi' = \frac{1 - \frac{\eta^2}{1 - c^2}}{\xi^2 + \frac{\eta^2}{1 - c^2}}.$$

Again, it may be written in the form

$$\left\{ \frac{\eta \cos \theta}{(1 - c^2)^{\frac{1}{2}}} - 1 \right\} = \xi^2 (1 - \cos^2 \theta);$$

$$\therefore \cos \phi \cdot \cos \phi' = \frac{1 - \xi^2}{\xi^2 + \frac{\eta^2}{1 - c^2}};$$

* Another solution of this and of several cognate problems, will be found in a paper by Mr. Hearn, in the *Cambridge and Dublin Mathematical Journal*, vol. iv. p. 265, entitled "Singular Application of Geometry of Three Dimensions to a Plane Problem."

therefore, by the given equation,

$$\cos \mu \left(\xi^2 + \frac{\eta^2}{1 - c^2} \right) = 1 - \xi^2 - \left(1 - \frac{\eta^2}{1 - c^2} \right) (1 - c^2 \sin^2 \mu)^{\frac{1}{2}},$$

$$\text{or } \xi^2 (1 + \cos \mu) + \frac{\eta^2}{1 - c^2} \{ \cos \mu - (1 - c^2 \sin^2 \mu)^{\frac{1}{2}} \} = 1 - (1 - c^2 \sin^2 \mu)^{\frac{1}{2}},$$

the equation to the locus of (ξ, η) which is therefore an ellipse, the squares of whose semiaxes are respectively

$$\frac{1 - (1 - c^2 \sin^2 \mu)^{\frac{1}{2}}}{1 + \cos \mu}; \quad (1 - c^2)^{\frac{1}{2}} \frac{1 - (1 - c^2 \sin^2 \mu)^{\frac{1}{2}}}{\cos \mu - (1 - c^2 \sin^2 \mu)^{\frac{1}{2}}};$$

$$\text{or } \frac{1 - \cos \mu - (1 - \cos \mu) (1 - c^2 \sin^2 \mu)^{\frac{1}{2}}}{\sin^2 \mu};$$

$$\frac{1 - c^2 \sin^2 \mu - \cos \mu - (1 - \cos \mu) (1 - c^2 \sin^2 \mu)^{\frac{1}{2}}}{\sin^2 \mu};$$

therefore, if c' be the distance from its centre to its focus,

$$\begin{aligned} c'^2 &= \frac{1 - \cos \mu - (1 - \cos \mu) (1 - c^2 \sin^2 \mu)^{\frac{1}{2}}}{\sin^2 \mu} \\ &\quad - \frac{1 - \cos \mu - c^2 \sin^2 \mu - (1 - \cos \mu) (1 - c^2 \sin^2 \mu)^{\frac{1}{2}}}{\sin^2 \mu} \\ &= c^2; \end{aligned}$$

$$\therefore c' = c,$$

whence the ellipses are confocal.

14. If x, y, z, w , are linear functions of the coordinates of any point, such that no three of the lines represented by the equations $x = 0, y = 0, z = 0, w = 0$, meet in a point, the equation $w + (yz)^{\frac{1}{2}} + (zx)^{\frac{1}{2}} + (xy)^{\frac{1}{2}} = 0$ is that of a curve of the fourth order having three double tangents, $x = 0, y = 0, z = 0$, and three double points, $y = z = w, z = x = w, x = y = w$. Shew also that the six points of contact of the double tangents lie in a conic section.

Where the line $x = 0$ meets the curve,

$$w + (yz)^{\frac{1}{2}} + (zx)^{\frac{1}{2}} + (xy)^{\frac{1}{2}} = 0 \dots \dots \dots (1);$$

$$\text{we have also } w + (yz)^{\frac{1}{2}} = 0, \quad \text{or } w^2 = yz \dots \dots \dots (2).$$

It hence appears that the line $x = 0$ meets (1) only in the points where it meets the conic (2), that is, in two points only. But (1) when rationalized takes the form

$$(w^2 - yz - zx - xy)^2 = 4xyz(x + y + z - 2w) \dots (3),$$

under which form we see that the curve represented by it is of the fourth order; therefore the line $x = 0$ must meet it in four real, coincident, or imaginary points. Therefore, either each of the points in which $x = 0$ meets (1) must be a double point, or $x = 0$ must be a double tangent to (1); for from considerations of symmetry it is clear that both points must be of the same nature. Also we see that $y = 0$, $z = 0$ stand in exactly the same relation to (1) as $x = 0$ does.

Now consider the conic whose equation is

$$w^2 - yz - zx - xy = 0 \dots (4).$$

From (1), (2) we see that all the points in which $x = 0$, $y = 0$, $z = 0$, meet (1) lie in this conic. Hence these points cannot be double points, for if they were, a curve of the second order would intersect a curve of the fourth order in twelve points (coinciding two by two), which is impossible. Therefore $x = 0$, $y = 0$, $z = 0$, are double tangents to (1).

Again, if in (3) we put $x = w$, it becomes

$$\{x(x - y - z) - yz\}^2 = 4xyz(y + z - x),$$

which can be reduced to

$$x(x - y - z) + yz = 0 \dots (5),$$

shewing that the line $x = w$ meets the given curve only in the points in which it meets the conic (5), that is in two points only. Hence, either $x = w$ is a double tangent, or it must meet the curve in two double points.

Now at the points where it meets the curve, we have, as may be seen from (5), $z = x = w$, $x = y = w$ respectively.

Hence, where the line $x = w$ meets the curve, it also meets either the line $y = w$ or $z = w$, and from considerations of symmetry, if $x = w$ touch the curve, $y = w$ and $z = w$ must do so likewise at the same points.

Now we have already shewn that if $x = w$ do not touch the curve (1), it must meet in two double points, and we have just now proved that if it *does* touch the curve, $y = w$ and $z = w$ touch it, each at one of the points where $x = w$ does. Therefore two tangents can be drawn at these points in different directions, therefore they must be double points. Hence, in either case, the points in question are double points, viz. $z = x = w$ and $x = y = w$; and similarly, it may be shewn that $y = z = w$ is a double point.

15. (α). Describe a circle when two tangents are given, and a point from which a pair of tangents drawn to the circle shall include a given angle.

(β). By means of the properties of reciprocal polars, or otherwise, construct a conic section, when the focus, two points, and the angle between the asymptotes are given.

(α). Let AB, AC (fig. 77) be the two tangents, P the point from which a pair of tangents are to include a given angle α . Let $\angle BAC = \beta$. Bisect the angle BAC by the straight line AD , join PA , and divide it in E , so that $PE : EA :: \sin \frac{1}{2}\beta : \sin \frac{1}{2}\alpha$. Also produce PA to F , so that $PF : AF :: \sin \frac{1}{2}\beta : \sin \frac{1}{2}\alpha$. Bisect EF in G , and with G as centre and GE as radius, describe a circle, cutting AD in H . H shall be the centre of the required circle.

For since $PE : EA :: PF : FA :: \sin \frac{1}{2}\beta : \sin \frac{1}{2}\alpha$, the locus of a point, the ratio of whose distances from P and A $= \sin \frac{1}{2}\beta : \sin \frac{1}{2}\alpha$ is the circle of which EF is a diameter. Therefore, joining PH , $PH : AH :: \sin \frac{1}{2}\beta : \sin \frac{1}{2}\alpha$.

Draw PK a tangent to the circle whose centre is H and which touches AB, AC , then if r be the radius of that circle,

$$\frac{r}{PH} = \sin HPK, \quad \frac{r}{AH} = \sin \frac{1}{2}\beta;$$

$$\therefore \sin HPK : \sin \frac{1}{2}\beta :: AH : PH \\ :: \sin \frac{1}{2}\alpha : \sin \frac{1}{2}\beta;$$

$$\therefore HPK = \frac{1}{2}\alpha,$$

and a pair of tangents to the circle drawn from P include the required angle α .

(β). If we take the polar reciprocal of the above system with respect to P , to the circle will correspond an hyperbola whose focus is P ; to the two given tangents AB, AC correspond two given points, and to the tangent through P , including a given angle, correspond two points on the curve at an infinite distance subtending a given angle at the focus or (since the points are infinitely distant) at the centre. This angle therefore is the angle between the asymptotes. Hence (β) is the polar reciprocal of (α); and therefore the required conic section may be constructed by means of the circle there determined.

16. Let P be any point in a conic section whose focus is S and eccentricity e ; in SP take $SQ = \frac{1}{2}L$ (the semi-latus-rectum); draw QR, ST perpendicular to SP , meeting the tangent at P in R and T respectively; also draw SY perpendicular to the tangent meeting it in Y ; and let PU, QZ drawn parallel to the transverse axis meet SY in U and Z respectively: it is required to prove one of the following properties:

- (1) R is a point in the latus-rectum.
- (2) QR passes through the point U .
- (3) $PU = e.PS$. (4) $SR = e.ST$. (5) $SY.SZ = (\frac{1}{2}L)^2$.

(1). Let the inclination of SP to the axis-major be α , then the polar equation to the tangent at P will be

$$\frac{1}{r} = \frac{2}{L} \{(\cos \theta + \cos (\theta - \alpha))\};$$

and that to QR , $r = \frac{1}{2}L \sec (\theta - \alpha)$.

At R , the point of intersection of these lines, we must have

$$\theta = \frac{1}{2}\pi;$$

therefore R is a point in the latus-rectum.

(2). The equation to SY is

$$\tan \theta = \frac{\sin \alpha}{e + \cos \alpha},$$

or in rectangular coordinates,

$$x \sin \alpha = y (e + \cos \alpha).$$

Now the ordinate of P is

$$\frac{\frac{1}{2}L \sin \alpha}{1 + e \cos \alpha};$$

therefore the equation to PU is

$$y = \frac{\frac{1}{2}L \sin \alpha}{1 + e \cos \alpha}.$$

At U the point of intersection of these, we have

$$x = \frac{\frac{1}{2}L (e + \cos \alpha)}{1 + e \cos \alpha},$$

$$\therefore x \cos \alpha + y \sin \alpha = \frac{1}{2}L.$$

Now this is the rectangular equation to QR ; hence QR passes through the point U .

$$(3). \text{ The abscissa of } P \text{ is } \frac{\frac{1}{2}L \cos \alpha}{1 + e \cos \alpha}.$$

That of U has been shewn to be

$$\frac{\frac{1}{2}L (e + \cos \alpha)}{1 + e \cos \alpha},$$

and P, U , have the same ordinate; hence

$$PU = e.PS.$$

(4). Since R is a point in the latus-rectum, SR is perpendicular to PU , and ST is perpendicular to SP , TR to SY , whence it readily follows that the triangles STR, SPU are similar;

$$\therefore PU : PS :: SR : ST;$$

$$\text{but } PU = e.PS,$$

$$\therefore SR = e.ST.$$

(5). The polar equation to QZ is

$$r \sin \theta = \frac{1}{2}L \sin \alpha,$$

and that to SY is

$$\tan \theta = \frac{\sin \alpha}{e + \cos \alpha};$$

$$\therefore \sin \theta = \frac{\sin \alpha}{(1 + 2e \cos \alpha + e^2)^{\frac{1}{2}}};$$

therefore at Z , the intersection of these lines,

$$r = SZ = \frac{1}{2} L (1 + 2e \cos \alpha + e^2)^{\frac{1}{2}}.$$

Also

$$SY = \frac{1}{2} L \frac{1}{\{(e + \cos \alpha)^2 + \sin^2 \alpha\}^{\frac{1}{2}}}$$

$$= \frac{1}{2} L \frac{1}{(1 + 2e \cos \alpha + e^2)^{\frac{1}{2}}};$$

$$\therefore SY.SZ = (\frac{1}{2} L)^2.$$

17. If $A_1, A_2 \dots A_n$; $a_1, a_2 \dots a_n$, be the angular points of two polygons of n sides each, which circumscribe a given circle, and $P_1, P_2 \dots P_n$ the points of intersection of their first, second... n^{th} sides respectively; shew that

$$P_1 A_1 . P_1 a_1 . P_2 A_2 . P_2 a_2 \dots P_n A_n . P_n a_n = P_1 A_2 . P_1 a_2 . P_2 A_3 . P_2 a_3 \dots P_n A_1 . P_n a_1.$$

Shew also, by means of projective properties or otherwise, that the same equation is true when any conic section is substituted for a circle.

From O , the centre of the circle, draw perpendiculars $OB_1, OB_2 \dots OB_n, Ob_1, Ob_2 \dots Ob_n$, on the sides $A_1 A_2, A_2 A_3 \dots A_n A_1, a_1 a_2, a_2 a_3 \dots a_{n-1} a_n$, respectively. Through O draw any line OX , and let generally $B_r OX = \alpha_r, b_r OX = \beta_r$ (fig. 78). Then

$$A_r OX = \frac{1}{2} (\alpha_{r-1} + \alpha_r), \quad \therefore A_r OB_r = A_r OB_{r-1} = \frac{1}{2} (\alpha_r - \alpha_{r-1}).$$

Similarly,

$$a_r OB_r = \frac{1}{2} (\beta_r - \beta_{r-1}),$$

$$\text{and } P_r OX = \frac{1}{2} (\alpha_r + \beta_r);$$

$$\therefore P_r OB_r = P_r OX - B_r OX = \frac{1}{2} (\beta_r - \alpha_r).$$

Also

$$P_r A_r = a (\tan A_r OB_r + \tan P_r OX_r),$$

a being the radius of the circle,

$$= a \{ \tan \frac{1}{2} (\alpha_r - \alpha_{r-1}) + \tan \frac{1}{2} (\beta_r - \alpha_r) \}$$

$$= a \frac{\sin \frac{1}{2} (\beta_r - \alpha_{r-1})}{\cos \frac{1}{2} (\alpha_r - \alpha_{r-1}) \cos \frac{1}{2} (\beta_r - \alpha_r)}.$$

Also
$$\begin{aligned} P_r A_{r+1} &= a (\tan A_{r+1} O B_r - \tan P_r O B_r) \\ &= a \{ \tan \frac{1}{2} (\alpha_{r+1} - \alpha_r) - \tan \frac{1}{2} (\beta_r - \alpha_r) \} \\ &= a \frac{\sin \frac{1}{2} (\alpha_{r+1} - \beta_r)}{\cos \frac{1}{2} (\alpha_{r+1} - \alpha_r) \cos \frac{1}{2} (\beta_r - \alpha_r)}. \end{aligned}$$

The expressions for $P_r a_r$ and $P_r a_{r+1}$, will be got from these by simply interchanging α and β ;

$$\begin{aligned} \therefore P_r a_r &= a \frac{\sin \frac{1}{2} (\alpha_r - \beta_{r-1})}{\cos \frac{1}{2} (\beta_r - \beta_{r-1}) \cos \frac{1}{2} (\alpha_r - \beta_r)}, \\ P_r a_{r+1} &= a \frac{\sin \frac{1}{2} (\beta_{r+1} - \alpha_r)}{\cos \frac{1}{2} (\beta_{r+1} - \beta_r) \cos \frac{1}{2} (\alpha_r - \beta_r)}. \end{aligned}$$

Hence
$$\frac{P_r A_r \cdot P_r a_r}{P_r A_{r+1} \cdot P_r a_{r+1}} = \frac{\sin \frac{1}{2} (\beta_r - \alpha_{r-1}) \sin \frac{1}{2} (\alpha_r - \beta_{r-1}) \cos \frac{1}{2} (\alpha_{r+1} - \alpha_r) \cos \frac{1}{2} (\beta_{r+1} - \beta_r)}{\sin \frac{1}{2} (\beta_{r+1} - \alpha_r) \sin \frac{1}{2} (\alpha_{r+1} - \beta_r) \cos \frac{1}{2} (\alpha_r - \alpha_{r-1}) \cos \frac{1}{2} (\beta_r - \beta_{r-1})} \dots (1).$$

From the form of this expression it is easy to see, that if we give r every value from 1 to n inclusive and multiply the resulting fractions together (observing that instead of α_{r+1} , β_{r+1} , we write $\alpha_1 \beta_1$), every factor will appear both in the numerator and denominator. Hence

$$\frac{P_1 A_1 \cdot P_1 a_1 \cdot P_2 A_2 \cdot P_2 a_2 \dots P_n A_n \cdot P_n a_n}{P_1 A_2 \cdot P_1 a_2 \cdot P_2 A_3 \cdot P_2 a_3 \dots P_n A_1 \cdot P_n a_1} = 1,$$

or
$$P_1 A_1 \cdot P_1 a_1 \cdot P_2 A_2 \cdot P_2 a_2 \dots P_n A_n \cdot P_n a_n = P_1 A_2 \cdot P_1 a_2 \cdot P_2 A_3 \cdot P_2 a_3 \dots P_n A_1 \cdot P_n a_1.$$

If for $P_r A_r$ we substitute $\frac{OP_r \cdot OA_r \cdot \sin P_r O A_r}{OB_r}$, and make similar substitution for each of the other lines, each member of this equation will, since $OB_r = a$, be divisible by

$$\frac{OA_1 \cdot OA_1 \cdot OP_1^2 \dots OA_n \cdot OA_n \cdot OP_n^2}{a^{2n}},$$

and there will remain merely a relation between the sines of angles subtended at O . The property just proved must there-

fore be true for any figure into which the circle can be projected, that is for any conic section. (See the article on the Method of Projections, in Salmon's *Conic Sections*, Chap. XIV.)

18. Prove one of the two following properties.

(1). When one of the foci of a conic section and two tangents are given, the locus of the other focus is a straight line.

(2). When the centre of the conic section and two tangents are given, the locus of the focus is an equilateral hyperbola.

The proof of these theorems depends on the property, that the product of the perpendiculars from the foci on the tangent at any point of a conic section is constant and equal to the square of the semiaxis minor.

(1). Let AP, AQ (fig. 79) be the two given tangents, S the given focus, H that whose locus is to be found. Draw SY, HZ perpendicular to AP ; SY', HZ' to AQ ; then

$$SY.HZ = SY'.HZ',$$

$$\therefore HZ : HZ' :: SY' : SY,$$

a constant ratio; therefore the locus of H is a straight line.

(2). Take the centre as origin, and let the equations to the given tangents be

$$x \cos \alpha + y \sin \alpha - a = 0 \quad (1), \quad x \cos \alpha' + y \sin \alpha' - a' = 0 \quad (2).$$

Let ξ, η , be the coordinates of one focus, then $-\xi, -\eta$, will be those of the other. Now the length of the perpendicular from ξ, η , to (1) is

$$\xi \cos \alpha + \eta \sin \alpha - a;$$

similarly, that from $-\xi, -\eta$, is

$$-\xi \cos \alpha - \eta \sin \alpha - a.$$

Hence we get

$$(\xi \cos \alpha + \eta \sin \alpha)^2 - a^2 = \beta^2,$$

β being the semiaxis minor.

Similarly it may be shewn that

$$(\xi \cos \alpha' + \eta \sin \alpha')^2 - a'^2 = \beta^2;$$

$$\therefore (\xi \cos \alpha + \eta \sin \alpha)^2 - \alpha^2 = (\xi \cos \alpha' + \eta \sin \alpha')^2 - \alpha'^2,$$

$$\text{or } (\xi \cos \alpha + \eta \sin \alpha)^2 - (\xi \cos \alpha' + \eta \sin \alpha')^2 = \alpha^2 - \alpha'^2,$$

the equation to the locus of ξ, η , which is therefore a rectangular hyperbola, the equations to whose asymptotes are

$$\xi (\cos \alpha + \cos \alpha') + \eta (\sin \alpha + \sin \alpha') = 0,$$

$$\xi (\cos \alpha - \cos \alpha') + \eta (\sin \alpha - \sin \alpha') = 0.$$

DIFFERENTIAL CALCULUS.

1848.

Find the equation to that involute of a cycloid which passes through the cusp, and shew that in the immediate neighbourhood of the cusp it becomes the curve $2a(4y)^3 = (3x)^4$, a being the radius of the generating circle.

Let x, y be the coordinates of any point in the cycloid referred to the cusp as origin, and base as axis of x , s its distance measured along the arc from the cusp; ξ, η those of the corresponding point in the involute. The equation to the cycloid will be

$$\begin{aligned}x &= a(\theta - \sin \theta), \\y &= a(1 - \cos \theta),\end{aligned}$$

and we have, since the tangent at (xy) passes through $(\xi\eta)$ at a distance s from (xy) ,

$$\begin{aligned}\xi &= x - s \frac{dx}{ds}, \\ \eta &= y - s \frac{dy}{ds}.\end{aligned}$$

$$\text{Now } \frac{dx}{d\theta} = a(1 - \cos \theta) = 2a \sin^2 \frac{1}{2}\theta,$$

$$\frac{dy}{d\theta} = a \sin \theta = 2a \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta;$$

$$\therefore \frac{ds}{d\theta} = 2a \sin \frac{1}{2}\theta,$$

$$\text{and } s = 4a(1 - \cos \frac{1}{2}\theta);$$

$$\therefore \xi = a(\theta - \sin \theta) - 4a(1 - \cos \frac{1}{2}\theta) \sin \frac{1}{2}\theta,$$

$$= a(\theta + \sin \theta - 4 \sin \frac{1}{2}\theta),$$

$$\eta = a(1 - \cos \theta) - 4a(1 - \cos \frac{1}{2}\theta) \cos \frac{1}{2}\theta,$$

$$= a(3 + \cos \theta - 4 \cos \frac{1}{2}\theta),$$

the equations to the involute.

In the immediate neighbourhood of the origin where θ is small, these become

$$\begin{aligned}\xi &= a \left\{ \theta + \theta - \frac{\theta^3}{6} - 4 \left(\frac{\theta}{2} - \frac{\theta^3}{6.8} \right) \right\}, \\ &= -a \frac{\theta^3}{12}.\end{aligned}$$

$$\begin{aligned}\text{and } \eta &= a \left\{ 1 - \frac{\theta^2}{2} + \frac{\theta^4}{2.4} - 4 \left(1 - \frac{\theta^2}{2.4} + \frac{\theta^4}{2.4.16} \right) \right\}, \\ &= a \frac{\theta^4}{8.4}.\end{aligned}$$

Hence, eliminating θ ,

$$\begin{aligned}\left(\frac{2^5 \eta}{a} \right)^3 &= \theta^{12} = \left(\frac{12 \xi}{a} \right)^4, \\ \text{or } 2a(4\eta)^3 &= (3\xi)^4,\end{aligned}$$

the required curve.

1849.

1. If P be a point in a cycloid, and O the corresponding position of the centre of the generating circle, shew that PO touches another cycloid of half the dimensions.

Let $a(\theta - \sin \theta)$, $a(1 - \cos \theta)$ be the coordinates of P , as in the last problem; then $a\theta$ and a will be those of O .

The equation to PO is

$$\frac{x - a\theta}{\sin \theta} + \frac{y - a}{\cos \theta} = 0,$$

$$\text{or } x \cos \theta + y \sin \theta = a(\theta \cos \theta + \sin \theta).$$

Differentiating this equation with respect to θ as variable parameter,

$$-x \sin \theta + y \cos \theta = a(2 \cos \theta - \theta \sin \theta);$$

$$\therefore x = a(\theta - \sin \theta \cos \theta),$$

$$= \frac{1}{2}a(2\theta - \sin 2\theta) \dots \dots \dots (1),$$

$$\text{and } y = a(1 + \cos^2 \theta)$$

$$= \frac{1}{2}a(1 + \cos 2\theta) \dots \dots \dots (2).$$

Equations (1) and (2) shew that the line OP always touches a cycloid whose cusp coincides with the cusp of the original cycloid, and generated by a circle of half the size of its generating circle.

2. Find the locus of the ultimate intersections of the lines defined by the equation

$$x \cos 3\theta + y \sin 3\theta = a(\cos 2\theta)^{\frac{3}{2}} \dots\dots\dots (1),$$

where θ is the variable parameter.

Differentiating (1) with respect to θ ,

$$x \sin 3\theta - y \cos 3\theta = a \sin 2\theta (\cos \theta)^{\frac{1}{2}} \dots\dots\dots (2).$$

Squaring (1) and (2), and adding,

$$x^2 + y^2 = a^2 \cos 2\theta \dots\dots\dots (3).$$

Again, (2) \div (1) gives

$$\frac{x \sin 3\theta - y \cos 3\theta}{x \cos 3\theta + y \sin 3\theta} = \tan 2\theta,$$

$$\text{or } \frac{\tan 3\theta - \frac{y}{x}}{1 + \tan 3\theta \frac{y}{x}} = \tan 2\theta;$$

$$\therefore \frac{y}{x} = \tan \theta,$$

$$\text{and by (3) } x^2 + y^2 = a^2 \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta},$$

$$= a^2 \frac{x^2 - y^2}{x^2 + y^2},$$

$$\text{or } (x^2 + y^2)^2 = a^2 (x^2 - y^2),$$

the equation to Bernoulli's Lemniscate.

3. If e be the eccentricity of a conic section, r the distance of any point from the focus, ρ the radius of curvature at that point, and ds an element of the arc of the curve, then

$$e^2 = \frac{dr^2}{ds^2} + \rho^2 \left(\frac{d^2 r}{ds^2} \right)^2.$$

Let the equation to the conic section referred to its focus as origin and axis-major as axis of x , be

$$x^2 + y^2 = (ex + c)^2;$$

$$\therefore r = ex + c,$$

$$\text{and } \frac{dr}{ds} = \frac{\frac{dr}{dx}}{\frac{ds}{dx}} = \frac{e}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}};$$

$$\therefore \frac{d^2r}{ds^2} = \frac{d}{dx} \frac{dr}{ds} \cdot \frac{1}{\frac{ds}{dx}},$$

$$= - \frac{e \frac{d^2y}{dx^2} \frac{dy}{dx}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} \cdot \frac{1}{\frac{ds}{dx}},$$

$$= - \frac{e}{\rho} \frac{\frac{dy}{dx}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}};$$

$$\therefore \left(\frac{dr}{ds}\right)^2 + \rho^2 \left(\frac{d^2r}{ds^2}\right)^2 = e^2 \frac{1 + \left(\frac{dy}{dx}\right)^2}{1 + \left(\frac{dy}{dx}\right)^2},$$

$$= e^2.$$

4. If u be a function of the independent variables x, y, z , given by the equations

$$u = f(s, t) \dots \dots \dots (1),$$

$$s = F(lx + my + nz + kt)$$

$$= \phi(mz - ny) + \chi(nx - lz) + \psi(ly - mx),$$

and if $l^2 + m^2 + n^2 = k^2$; shew that

$$l \frac{du}{dx} + m \frac{du}{dy} + n \frac{du}{dz} + k \frac{du}{dt} = 0,$$

where $\frac{du}{dt}$ is obtained from (1) by considering s constant.

We have

$$\frac{du}{dx} = f'(s) \frac{ds}{dx} + f'(t) \frac{dt}{dx},$$

$$\text{and } \frac{ds}{dx} = \left(l + k \frac{dt}{dx} \right) F',$$

$$\text{also } = n\chi' - m\psi';$$

$$\therefore \frac{dt}{dx} = \frac{n\chi' - m\psi'}{kF'} - \frac{l}{k},$$

$$\therefore \frac{du}{dx} = \left\{ f'(s) + \frac{f'(t)}{kF'} \right\} (n\chi' - m\psi') - \frac{l}{k} f'(t);$$

$$\text{similarly } \frac{du}{dy} = \left\{ f'(s) + \frac{f'(t)}{kF'} \right\} (l\psi' - n\phi') - \frac{m}{k} f'(t),$$

$$\frac{du}{dz} = \left\{ f'(s) + \frac{f'(t)}{kF'} \right\} (m\phi' - l\chi') - \frac{m}{k} f'(t).$$

Multiplying these equations by l , m , n respectively, and adding, remembering that $l^2 + m^2 + n^2 = k^2$,

$$l \frac{du}{dx} + m \frac{du}{dy} + n \frac{du}{dz} + k \frac{du}{dt} = 0,$$

since $\frac{du}{dt}$ means $f'(t)$.

1850.

1. A paraboloid of revolution with its axis vertical contains a quantity of water, into which is sunk a heavy sphere, and the water is just sufficient to cover the sphere; find the form of the paraboloid that the quantity of water with which this can be done may be the least possible.

Let a be the radius of the sphere, l the latus-rectum of the paraboloid; h the height to which the water rises when the sphere is sunk: then if C be the content of the paraboloid of height h , V the volume of the sphere, Q the quantity of water,

$$Q = C - V;$$

and we have to make Q a minimum by the variation of l and h .

Now

$$C = \frac{1}{2} \pi l h^2,$$

$$V = \frac{4}{3} \pi a^3,$$

therefore lh^2 must be a minimum.

Now from the vertex the equation to the section of the paraboloid is

$$y^2 = lx;$$

that to the section of the sphere is

$$\{x - (h - a)\}^2 + y^2 = a^2.$$

In order that these may touch one another, we must have

$$\{x - (h - a)\}^2 + lx - a^2,$$

a perfect square, which requires that

$$4 \{h^2 - 2al\} = \{l - 2(h - a)\}^2,$$

$$\text{or } l^2 - 4l(h - a) + 4a^2 = 0 \dots\dots\dots (1).$$

Hence we must make lh^2 a minimum subject to the condition (1), which may be written

$$h = \frac{(l + 2a)^2}{4l};$$

therefore we have to make

$$\frac{(l + 2a)^4}{l} = \text{minimum},$$

$$\text{or } \frac{4}{l + 2a} - \frac{1}{l} = 0;$$

$$\therefore l = \frac{2}{3}a;$$

which determines the form of the paraboloid.

2. If a circle be described touching a curve at any point (r, θ) and passing through the pole, shew that the equation to the circle will be

$$r' + r^2 \frac{d}{d\theta} \frac{\sin(\theta' - \theta)}{r} = 0.$$

The general equation to a circle passing through the pole and the point (r, θ) is

$$r' = r \sec(\theta - \alpha) \cos(\theta' - \alpha) \dots\dots\dots (1),$$

α being the angular coordinate of the diameter through the pole. This equation may be put in the form

$$\begin{aligned} r' &= r \frac{\cos \{\theta - \alpha + (\theta' - \theta)\}}{\cos (\theta - \alpha)} \\ &= r \{\cos (\theta' - \theta) - \tan (\theta - \alpha) \sin (\theta' - \theta)\}. \end{aligned}$$

Now from (1), $\frac{dr'}{d\theta'} = -r \sec (\theta - \alpha) \sin (\theta' - \alpha)$

$$= \frac{dr}{d\theta}, \text{ when } \theta' = \theta,$$

since the circle touches the curve at the point (r, θ) ;

$$\therefore r \tan (\theta - \alpha) = \frac{dr}{d\theta},$$

$$\begin{aligned} \text{and } r' &= r \{\cos (\theta' - \theta) + \frac{dr}{d\theta} \sin (\theta' - \theta)\} \\ &= -r^2 \left\{ \frac{1}{r} \frac{d}{d\theta} \sin (\theta' - \theta) + \sin (\theta' - \theta) \frac{d}{d\theta} \frac{1}{r} \right\} \\ &= -r^2 \frac{d}{d\theta} \frac{\sin (\theta' - \theta)}{r}, \\ \text{or } r' + r^2 \frac{d}{d\theta} \frac{\sin (\theta' - \theta)}{r} &= 0. \end{aligned}$$

3. If a parabola roll upon a line, the focus will trace out a catenary.

The following more general problem admits of very easy solution: A given curve rolls upon a straight line, to find the locus of any point to which the curve is referred as pole.

Let AB (fig. 79) be the given straight line, A any fixed point in it. Let CP be the rolling curve, C the point which has been in contact with A , S the pole, P the point of contact in the position represented in the figure. Join SP and draw SY perpendicular to AB . Let $AY = x$, $YS = y$, $SP = r$, s the arc of the curve described by S . Then the tangent being manifestly perpendicular to SP , we have

$$\frac{dx}{ds} = \cos PSY = \frac{y}{r},$$

* For this solution, we are indebted to Mr. Goodwin

and $y^2 \left(\frac{ds}{dx} \right)^2$ or $y^2 \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\} = r^2$.

Let the equation to the rolling curve be $r^2 = f(p)$, then the equation of the required locus is

$$y^2 \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\} = f(y).$$

In the case of the parabola, we have

$$r = \frac{p^2}{c};$$

$$\therefore \frac{dx}{ds} = \frac{y}{r} = \frac{c}{y},$$

the differential equation to the catenary.

4. Find the forms of the curve whose equation is

$$xy^3 = m^2 (x^2 + y^2 - a^2),$$

according as m^2 is $>$ or $<$ $\frac{3 \cdot 3^{\frac{1}{2}}}{4} a^2$.

Arranging the equation as a quadratic in x , we have

$$x^2 - \frac{y^3}{m^2} \cdot x = a^2 - y^2,$$

and
$$x = \frac{y^3}{2m^2} \pm \frac{1}{2m^2} \{y^6 + 4m^4 (a^2 - y^2)\}^{\frac{1}{2}} \dots\dots\dots (1).$$

Hence we may use $x = \frac{y^3}{2m^2}$ as a guiding curve; its form is shewn by the dotted curve.

To consider the equation

$$y^6 + 4m^4 (a^2 - y^2) = 0,$$

$$\text{or } y^6 - 4m^4 \cdot y^2 + 4m^4 a^2 = 0.$$

This equation, considered as a cubic in y^2 , will have three real roots or one, according as m^2 is $>$ or $<$ $\frac{3 \cdot 3^{\frac{1}{2}}}{4} a^2$.* If it has three real roots, one of them is negative, and the corre-

* If $m = \frac{3 \cdot 3^{\frac{1}{2}}}{4} a^2$, it will have three real roots, two of them being equal.

sponding values of y imaginary. If it has two equal, and therefore two real roots, the equal roots are positive and the other negative, giving only one positive value to y^2 . If it has only one real root, it is negative and y imaginary. Again, differentiating the original equation, we get

$$y^3 + 3y^2x \frac{dy}{dx} = m^2 \left(2x + 2y \frac{dy}{dx} \right);$$

$$\therefore \frac{dy}{dx} = \frac{2m^2x - y^3}{3y^2x - 2m^2y};$$

therefore, when $y = 0$, and therefore $x = a$,

$$\frac{dy}{dx} = \infty.$$

Also, taking the lower sign in equation (1),

$$x = \frac{y^3}{2m^2} - \frac{y^3}{2m^2} \left(1 + 4m^4 \frac{a^2 - y^2}{y^6} \right)^{\frac{1}{2}}$$

$$= -m^2 \frac{a^2 - y^2}{y^3} - \&c.$$

$$= 0 \text{ if } y = \infty:$$

hence the axis of y is an asymptote to the curve, which we thus see has the form represented in figs. 80, 81, 82, according as m^2 is $> =$ or $< \frac{3.3^{\frac{1}{2}}}{4} a^2$.

5. Trace the curve whose equation is

$$(r - c)^2 = c\theta (2a - c\theta) \text{ when } c = \frac{2a}{\pi},$$

and prove that as c increases indefinitely the curve approximates to a circle.

When θ is positive, $c\theta$ must be less than $2a$ or θ less than π , and θ can receive no negative value. Also, any value θ_1 of θ gives the same value for r as $\pi - \theta_1$. Solving the equation we have

$$\begin{aligned} r &= c \pm \{c\theta (2a - c\theta)\}^{\frac{1}{2}} \\ &= \frac{2a}{\pi} \pm 2a \left\{ \frac{\theta}{\pi} \left(1 - \frac{\theta}{\pi} \right) \right\}^{\frac{1}{2}} \dots\dots\dots (1). \end{aligned}$$

The quantity affected with the ambiguity has its greatest value when $\frac{\theta}{\pi} = \frac{1}{2}$, when r receives the value

$$r = \frac{2a}{\pi} \pm a,$$

the latter of which is negative. Again, putting $\theta = 0$, we have $r = c$; and differentiating,

$$2(r - c) \frac{dr}{d\theta} = c(2a - c\theta) - c^2\theta;$$

therefore when $r = c$, and therefore $\theta = 0$,

$$\frac{dr}{d\theta} = \infty.$$

Hence the curve is of the form shewn in fig. 83.

Also, as c and therefore a is indefinitely enlarged, equation (1) becomes $r = \frac{2a}{\pi}$, representing a circle.

6. Find the locus of the consecutive intersections of the curve whose equation is $x'' + y'' = 2ax' + 2by'$ (1); a and b having any values which satisfy the equations

$$2a \left(x \frac{dy}{dx} - y \right) = (x^2 - y^2) \frac{dy}{dx} - 2xy \dots\dots\dots (2),$$

$$2b \left(x \frac{dy}{dx} - y \right) = x^2 - y^2 + 2xy \frac{dy}{dx} \dots\dots\dots (3),$$

x and y being the coordinates of any given curve.

The problem is best solved by introducing polar coordinates.

Let $x = r \cos \theta$, $y = r \sin \theta$;

$$\therefore dx = -r \sin \theta d\theta + \cos \theta dr, \quad dy = r \cos \theta d\theta + \sin \theta dr;$$

$$\therefore xdy - ydx = r^2 d\theta,$$

$$\begin{aligned} (x^2 - y^2) dy - 2xy dx &= r^2 \cos 2\theta dy - r^2 \sin 2\theta dx \\ &= r^3 \cos \theta d\theta - r^2 \sin \theta dr, \end{aligned}$$

$$(x^2 - y^2) dx + 2xy dy = r^3 \sin \theta d\theta + r^2 \cos \theta dr.$$

Hence equations (2) and (3) become

$$2a = r \cos \theta - \sin \theta \frac{dr}{d\theta},$$

$$2b = r \sin \theta + \cos \theta \frac{dr}{d\theta};$$

$$\therefore 2 \frac{da}{d\theta} = -r \sin \theta - \sin \theta \frac{d^2r}{d\theta^2},$$

$$2 \frac{db}{d\theta} = r \cos \theta - \cos \theta \frac{d^2r}{d\theta^2}.$$

Also equation (1) transformed into polar coordinates becomes

$$r' = 2a \cos \theta' + 2b \sin \theta'.$$

Differentiating this equation with respect to θ , considering r' and θ' constant, we have

$$\begin{aligned} 0 &= 2 \frac{da}{d\theta} \cos \theta' + 2 \frac{db}{d\theta} \sin \theta' \\ &= r \sin (\theta' - \theta) - \cos (\theta' - \theta) \frac{d^2r}{d\theta^2}; \\ \therefore \tan (\theta' - \theta) &= \frac{\frac{d^2r}{d\theta^2}}{r}. \end{aligned}$$

From this equation, when r has been substituted in terms of θ , from the known equation to the curve, we can find θ in terms of θ' ; and thence r and $\frac{dr}{d\theta}$ will be known in terms of θ' , and the required equation to the curve will be

$$\begin{aligned} r' &= 2a \cos \theta' + 2b \sin \theta' \\ &= r \cos (\theta' - \theta) + \sin (\theta' - \theta) \frac{dr}{d\theta}. \end{aligned}$$

1851.

If $\phi(c)$ be a rational and integral function of c , the coefficients of which are functions of any number of variables x, y, \dots then if δ denote differentiation with respect to the variables, and the quantity c be eliminated from the equations $\phi(c) = 0$,

$\delta\phi(c) = 0$, the result may be represented by $u\delta P\delta Q\dots = 0$ where P, Q, \dots are the roots of the equation $\phi(c) = 0$, and $u = 0$ is the result of the elimination of c from the equations $\phi(c) = 0$, $\phi'(c) = 0$.

Since $\phi(c)$ is a rational and integral function of c , and P, Q, \dots are the roots of the equation $\phi(c) = 0$, we have

$$\phi(c) = (c - P)(c - Q)\dots\dots\dots \textit{identically}.$$

Let P_x denote $\frac{dP}{dx} dx$, $P_y, \frac{dP}{dy} dy, \dots$ then

$$\begin{aligned}\delta\phi(c) &= \phi(c) \delta\log\phi(c) \\ &= -\phi(c) \left(\frac{P_x + P_y + \dots}{c - P} + \frac{Q_x + Q_y + \dots}{c - Q} + \dots \right) \\ &= -\{(P_x + P_y + \dots)(c - Q)(c - R)\dots + (Q_x + Q_y + \dots)(c - P)(c - R) + \dots\}.\end{aligned}$$

Hence the result of the elimination of c between $\phi(c) = 0$ and $\delta\phi(c) = 0$, is

$$\begin{aligned}0 &= \text{product of the expressions } (P_x + P_y + \dots) (P - Q) (P - R) \dots, \\ &\quad (Q_x + Q_y + \dots) (Q - P) (Q - R) \dots, \text{ \&c.} \\ &= (P_x + P_y + \dots)(Q_x + Q_y + \dots)\dots(P - Q)(P - R)\dots(Q - P)(Q - R)\dots \\ &= v \text{ suppose.}\end{aligned}$$

$$\text{Again, } \phi'(c) = (c - Q)(c - R)\dots + (c - P)(c - R)\dots + \dots;$$

$$\therefore u = (P - Q)(P - R)\dots(Q - P)(Q - R)\dots,$$

$$\text{and } \delta P = P_x + P_y + \dots, \quad \delta Q = Q_x + Q_y + \dots;$$

$$\therefore v = u\delta P\delta Q\dots$$

Hence the result of the elimination may be represented by $u\delta P\delta Q\dots = 0$.

INTEGRAL CALCULUS.

1848.

THE corner of a sheet of paper is turned down so that the sum of the edges turned down is constant; find the equation to the curve traced out by the vertex of the angle; find also the area of the curve.

Let r, θ , be the polar coordinates of the vertex, referred to the original position of the vertex as pole, then the lengths of the respective edges are

$$\frac{1}{2}r \sec \theta, \quad \frac{1}{2}r \operatorname{cosec} \theta, \quad \text{respectively};$$

therefore the equation to the curve, is

$$\frac{1}{2}r (\sec \theta + \operatorname{cosec} \theta) = \text{constant} = a \text{ suppose,}$$

or in rectangular coordinates,

$$(x + y) (x^2 + y^2) = 2axy.$$

To find the area, turn the axes through an angle $\frac{1}{4}\pi$, then we get

$$r = \frac{a \cos 2\theta}{2^{\frac{1}{2}} \cos \theta};$$

therefore if A be the area of the loop traced out by the vertex,

$$\begin{aligned} A &= \frac{1}{2} \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} r^2 d\theta \\ &= \frac{a^2}{4} \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \frac{\cos^2 2\theta}{\cos^2 \theta} d\theta \\ &= \frac{a^2}{4} \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \left(4 \cos^2 \theta - 4 + \frac{1}{\cos^2 \theta} \right) d\theta \\ &= \frac{a^2}{4} \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} (2 \cos 2\theta - 2 + \sec^2 \theta) d\theta \\ &= \frac{a^2}{4} (2 - \pi + 2) \\ &= \left(1 - \frac{1}{4}\pi \right) a^2. \end{aligned}$$

2. Tangents to a system of similar and concentric ellipses are drawn at a given perpendicular distance from the centre; find the locus of the point of contact, and shew that the area of the curve is equal to that of an ellipse which has the same greatest and least diameters.

Take the common axes of the ellipses as coordinate axes, and let c be the distance of each tangent from the centre, then if θ be the inclination of a perpendicular on any tangent from the centre to the axis of x , the equation to that tangent will be

$$x \cos \theta + y \sin \theta = c \dots\dots\dots (1).$$

Let the equation to any one of the ellipses be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = m^2 \dots\dots\dots (2).$$

If (ξ, η) be the coordinates of the point where (1) touches this, we have

$$\frac{\xi}{m^2 a^2} = \frac{\cos \theta}{c}, \quad \frac{\eta}{m^2 b^2} = \frac{\sin \theta}{c};$$

and since ξ, η , is a point in (2),

$$m^2 = \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2}.$$

Eliminating m, θ between these equations, we get

$$\frac{\xi^2}{a^4} + \frac{\eta^2}{b^4} = \frac{1}{c^2} \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} \right)^2,$$

the equation to the locus of the point of contact.

To find the area of this curve, transform its equation to polar coordinates. It then becomes

$$r^2 = c^2 \frac{\frac{\cos^2 \theta}{a^4} + \frac{\sin^2 \theta}{b^4}}{\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)^2};$$

and if A be its area,

$$A = 2c^2 \int_0^{\frac{1}{2}\pi} \frac{\frac{\cos^2 \theta}{a^4} + \frac{\sin^2 \theta}{b^4}}{\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)^2} d\theta.$$

$$\text{Let } \frac{a}{b} \tan \theta = \tan \phi;$$

$$\therefore \frac{a}{b} \sec^2 \theta d\theta = \sec^2 \phi d\phi,$$

$$\begin{aligned} \text{and } A &= 2c^2 \int_0^{\frac{1}{2}\pi} \frac{1 + \frac{a^2}{b^2} \tan^2 \phi}{\sec^4 \phi} \frac{b}{a} \sec^2 \phi d\phi \\ &= \frac{2c^2}{ab} \int_0^{\frac{1}{2}\pi} (a^2 \sin^2 \phi + b^2 \cos^2 \phi) d\phi \\ &= \frac{1}{2} \pi \frac{c^2}{ab} (a^2 + b^2). \end{aligned}$$

Again, let A' be the area of the ellipse which has the same greatest and least diameters, then if these diameters be $2r_1, 2r_2$,

$$A' = \pi r_1 r_2.$$

$$\begin{aligned} \text{Now } r^2 &= c^2 \frac{\frac{\cos^2 \theta}{a^4} + \frac{\sin^2 \theta}{b^4}}{\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}\right)^2} \\ &= c^2 \frac{1 + \frac{a^2}{b^2} \tan^2 \phi}{\sec^4 \phi} \left(1 + \frac{b^2}{a^2} \tan^2 \phi\right) \\ &= \frac{c^2}{a^2 b^2} (a^2 \cos^2 \phi + b^2 \sin^2 \phi) (a^2 \sin^2 \phi + b^2 \cos^2 \phi) \\ &= \frac{c^2}{a^2 b^2} \left(\frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos 2\phi\right) \left(\frac{a^2 + b^2}{2} - \frac{a^2 - b^2}{2} \cos 2\phi\right) \\ &= \frac{c^2}{a^2 b^2} \left\{ \left(\frac{a^2 + b^2}{2}\right)^2 - \left(\frac{a^2 - b^2}{2}\right)^2 \cos^2 2\phi \right\}. \end{aligned}$$

The maximum and minimum values of r will be got by putting $\cos 2\phi = 0$ and 1 successively;

$$\therefore r_1 = \frac{c}{ab} \frac{a^2 + b^2}{2}, \quad r_2 = c,$$

$$\therefore A' = \frac{\pi c^2}{ab} \frac{a^2 + b^2}{2},$$

the same value as that previously got for A .

3. Prove that the remainder after n terms of the infinite series $\frac{1}{1^\alpha} + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \dots$ where $\alpha > 1$ lies between $\frac{1}{(\alpha-1)(n+\frac{1}{2})^{\alpha-1}}$, $\left(= \int_{n+\frac{1}{2}}^{\infty} \frac{dx}{x^\alpha} \right)$ and $\frac{1}{(\alpha-1)(n+1)^{\alpha-1}}$, approaching much nearer to the former limit when n is large.

(1). In general

$$\frac{1}{(m-p)^\alpha} + \frac{1}{(m+p)^\alpha} = 2 \frac{m^\alpha + \frac{\alpha(\alpha-1)}{1.2} m^{\alpha-2} p^2 + \dots}{(m^2 - p^2)^\alpha},$$

$$> \frac{2}{m^\alpha}.$$

Now

$$\begin{aligned} & \frac{1}{(n+1-rp)^\alpha} + \frac{1}{\{n+1-(r-1)p\}^\alpha} + \dots + \frac{1}{(n+1)^\alpha} + \dots \\ & \quad + \frac{1}{\{n+1+(r-1)p\}^\alpha} + \frac{1}{(n+1+rp)^\alpha}, \\ &= \left\{ \frac{1}{(n+1-rp)^\alpha} + \frac{1}{(n+1+rp)^\alpha} \right\} + \left[\frac{1}{\{n+1-(r-1)p\}^\alpha} \right. \\ & \quad \left. + \frac{1}{\{n+1+(r-1)p\}^\alpha} \right] + \dots + \frac{1}{(n+1)^\alpha}, \\ &> \frac{2}{(n+1)^\alpha} + \frac{2}{(n+1)^\alpha} + \dots + \frac{1}{(n+1)^\alpha}, \text{ from above,} \\ &> \frac{2r+1}{(n+1)^\alpha}; \\ &\therefore \frac{p}{(n+1-rp)^\alpha} + \frac{p}{\{n+1-(r-1)p\}^\alpha} + \dots > \frac{2rp+p}{(n+1)^\alpha}. \end{aligned}$$

Now let $rp = \frac{1}{2} - p$, then this becomes

$$\frac{p}{(n+\frac{1}{2}+p)^\alpha} + \frac{p}{(n+\frac{1}{2}+2p)^\alpha} + \dots + \frac{p}{(n+\frac{3}{2}-p)^\alpha} > \frac{1}{(n+1)^\alpha};$$

therefore, *à fortiori*,

$$\frac{p}{(n+\frac{1}{2}+p)^\alpha} + \dots + \frac{p}{(n+\frac{3}{2})^\alpha} > \frac{1}{(n+1)^\alpha}.$$

Let $p = dx$, then this becomes

$$\int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{dx}{x^x} > \frac{1}{(n+1)^x};$$

$$\text{similarly } \int_{n+\frac{3}{2}}^{n+\frac{5}{2}} \frac{dx}{x^x} > \frac{1}{(n+2)^x},$$

$$\dots\dots\dots > \dots\dots\dots$$

We shall thus obtain an infinite series of inequalities similar to the above. Adding them all together, we get

$$\int_{n+\frac{1}{2}}^{\infty} \frac{dx}{x^x} = \frac{1}{(\alpha-1)(n+\frac{1}{2})^{\alpha-1}} > \frac{1}{(n+1)^x} + \frac{1}{(n+2)^x} + \dots$$

Again, if $p = \frac{1}{p'}$, p' being any integer,

$$\begin{aligned} \frac{p}{(n+1+p)^x} + \frac{p}{(n+1+2p)^x} + \dots \quad (&p' \text{ terms}) \\ < \frac{p}{(n+1)^x} + \frac{p}{(n+1)^x} + \dots \quad (&p' \text{ terms}), \\ < \frac{1}{(n+1)^x}. \end{aligned}$$

For p write dx , then this becomes

$$\int_{n+1}^{n+2} \frac{dx}{x^x} < \frac{1}{(n+1)^x},$$

whence, as above, we get

$$\int_{n+1}^{\infty} \frac{dx}{x^x} = \frac{1}{(\alpha-1)(n+1)^{\alpha-1}} < \frac{1}{(n+1)^x} + \frac{1}{(n+2)^x} + \dots;$$

$$\text{We have } \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{dx}{x^x} = \frac{1}{(\alpha-1)} \left\{ \frac{1}{(n+\frac{1}{2})^{\alpha-1}} - \frac{1}{(n+\frac{3}{2})^{\alpha-1}} \right\},$$

$$= \frac{1}{(\alpha-1)(n+1)^{\alpha-1}} \left\{ \frac{1}{\left(1 - \frac{1}{2n+2}\right)^{\alpha-1}} - \frac{1}{\left(1 + \frac{1}{2n+2}\right)^{\alpha-1}} \right\},$$

$$\begin{aligned} &= \frac{2}{(\alpha-1)(n+1)^{\alpha-1}} \left\{ \frac{\alpha-1}{2n+2} \right. \\ &\quad \left. + \frac{(\alpha-1)(\alpha-2)(\alpha-3)}{6} \frac{1}{(2n+2)^3} + \dots \right\}, \end{aligned}$$

$$= \frac{1}{(n+1)^x} + \frac{(\alpha-2)(\alpha-3)}{24(n+1)^{x+2}} + \dots;$$

$$\therefore \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{dx}{x^\alpha} - \frac{1}{(n+1)^\alpha} = \frac{(\alpha-2)(\alpha-3)}{24(n+1)^{\alpha+2}} + \dots$$

$$\begin{aligned} \text{But } \int_{n+1}^{n+2} \frac{dx}{x^\alpha} &= \frac{1}{(\alpha-1)} \left\{ \frac{1}{(n+1)^{\alpha-1}} - \frac{1}{(n+2)^{\alpha-1}} \right\}, \\ &= \frac{1}{(\alpha-1)} \frac{1}{(n+1)^{\alpha-1}} \left\{ 1 - \frac{1}{\left(1 + \frac{1}{n+1}\right)^{\alpha-1}} \right\}, \\ &= \frac{1}{\alpha-1} \frac{1}{(n+1)^{\alpha-1}} \left\{ \frac{\alpha-1}{(n+1)} \frac{(\alpha-1)(\alpha-2)}{2} \frac{1}{(n+1)^2} + \dots \right\}, \\ &= \frac{1}{(n+1)^\alpha} - \frac{\alpha-2}{2} \frac{1}{(n+1)^{\alpha+1}} + \dots; \\ \therefore \frac{1}{(n+1)^\alpha} - \int_{n+1}^{n+2} \frac{dx}{x^\alpha} &= \frac{\alpha-2}{2(n+1)^{\alpha+1}} + \dots \end{aligned}$$

It hence appears that when n is large, $\frac{1}{(n+1)^\alpha}$ approaches much more nearly to the limit $\int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{dx}{x^\alpha}$ than to $\int_{n+1}^{n+2} \frac{dx}{x^\alpha}$, whence the latter part of the proposed theorem readily follows.

4. If $f(x)$ be positive and finite from $x = a$ to $x = a + h$, shew how to find the limit of

$$\left\{ f(a) f\left(a + \frac{1}{n} h\right) \dots f\left(a + \frac{n-1}{n} h\right) \right\}^{\frac{1}{n}},$$

for $n = \infty$, and prove that the limit in question is less than $\frac{1}{h} \int_a^{a+h} f(x) dx$, assuming that the geometric mean of a finite number of positive quantities which are not all equal is less than the arithmetic.

Hence prove that $\int_0^1 \varepsilon^{f(x)} dx < \int_0^1 \varepsilon^{u(x)} dx$, unless u be constant from $x = 0$ to $x = 1$.

Let $\log f(x) = F(x)$, then

$$\begin{aligned} &\log \left\{ f(a) f\left(a + \frac{1}{n} h\right) \dots f\left(a + \frac{n-1}{n} h\right) \right\}^{\frac{1}{n}}, \\ &= \frac{1}{n} \left\{ F(a) + F\left(a + \frac{1}{n} h\right) + \dots + F\left(a + \frac{n-1}{n} h\right) \right\}, \\ &= y \text{ suppose.} \end{aligned}$$

When n is infinite, let $\frac{h}{n} = dx$; then

$$\begin{aligned} y &= \frac{dx}{h} \{F(a) + F(a+dx) + \dots + F(a+h)\}, \\ &= \frac{1}{h} \int_a^h F(a+x) dx; \\ \therefore \left\{ f(a) f\left(a + \frac{1}{n} h\right) \dots f\left(a + \frac{n-1}{n} h\right) \right\}^{\frac{1}{n}} \end{aligned}$$

approaches to the limit $\varepsilon^{\frac{1}{h} \int_a^h \log f(a+x) dx}$ or $\varepsilon^{\frac{1}{h} \int_a^{a+h} \log f(x) dx}$. Now since the geometric mean of a finite number of positive quantities which are not all equal, is less than the arithmetic,

$$\begin{aligned} \left\{ f(a) f\left(a + \frac{1}{n} h\right) \dots f\left(a + \frac{n-1}{n} h\right) \right\}^{\frac{1}{n}} \\ < \frac{1}{n} \left\{ f(a) + f\left(a + \frac{1}{n} h\right) + \dots + f\left(a + \frac{n-1}{n} h\right) \right\}, \end{aligned}$$

as long as n is finite, and this will hold up to the limit when n is indefinitely increased: but in that case

$$\frac{1}{n} \left\{ f(a) + f\left(a + \frac{1}{n} h\right) + \dots + f\left(a + \frac{n-1}{n} h\right) \right\} = \frac{1}{h} \int_a^{a+h} f(x) dx;$$

therefore the required limit $< \frac{1}{h} \int_a^{a+h} f(x) dx$.

Hence if $f(x) = e^u$, $a = 0$, and $h = 1$, $\varepsilon^{\int_0^1 u dx} < \int_0^1 \varepsilon^u dx$, unless u be constant from $x = 0$ to $x = 1$, in which case they are equal.

1849.

1. Investigate the series

$$\frac{\theta^2}{4} = \frac{\pi^2}{12} - \cos \theta + \frac{\cos 2\theta}{2^2} - \frac{\cos 3\theta}{3^2} + \&c.$$

for values of θ between $-\pi$ and π .

$$\begin{aligned} \text{Let} \quad \cos \theta - \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta - \dots &= u, \\ \sin \theta - \frac{1}{2} \sin 2\theta + \frac{1}{3} \cos 3\theta - \dots &= v; \end{aligned}$$

then if θ lie between $-\pi$ and π ,

$$\begin{aligned} u + -\frac{1}{2}v &= \varepsilon^{-\frac{1}{2}\theta} - \frac{1}{2}\varepsilon^{-\frac{1}{2}2\theta} + \frac{1}{3}\varepsilon^{-\frac{1}{2}3\theta} - \dots, \\ &= \log(1 + \varepsilon^{-\frac{1}{2}\theta}) = \log(\varepsilon^{-\frac{1}{2}\theta} + \varepsilon^{-\frac{1}{2}\theta}) + \log \varepsilon^{-\frac{1}{2}\theta}, \\ &= \log(2 \cos \frac{1}{2}\theta) + -\frac{1}{2}\theta; \end{aligned}$$

therefore equating imaginary parts,

$$v = \sin \theta - \frac{1}{2} \sin 2\theta + \frac{1}{3} \cos 3\theta - \dots = \frac{1}{2}\theta;$$

integrating with respect to θ ,

$$-\cos \theta + \frac{1}{2^2} \cos 2\theta - \frac{1}{3^2} \cos 3\theta + \dots = \frac{\theta^2}{4} + C.$$

To determine the constant, put $\theta = 0$;

$$\therefore -1 + \frac{1}{2^2} - \frac{1}{3^2} + \dots = C.$$

$$\text{Now } 1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \dots\right),$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots - \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right),$$

$$= \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right):$$

$$\text{also } \sin \theta = \theta \left\{1 - \left(\frac{\theta}{\pi}\right)^2\right\} \left\{1 - \left(\frac{\theta}{2\pi}\right)^2\right\} \dots,$$

$$\text{and } \sin \theta = \theta - \frac{\theta^3}{1.2.3} + \dots:$$

equating coefficients of θ^3 ,

$$\frac{1}{\pi^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) = \frac{1}{6},$$

$$\therefore C = -\frac{1}{12}\pi^2,$$

$$\text{and } \frac{\theta^2}{4} = \frac{\pi^2}{12} - \cos \theta + \frac{1}{2^2} \cos 2\theta - \frac{1}{3^2} \cos 2\theta + \dots$$

2. If a line be drawn through the centre of an ellipse, cutting the major axis at an angle θ , and the curve at an angle

ϕ , (1) prove that

$$(a^2 - b^2) \cos(2\theta - \phi) = (a^2 + b^2) \cos \phi;$$

and (2) that $\int_0^\pi \phi d\theta = \frac{\pi^2}{2}$.

(1). Let the coordinates of the point where the straight line meets the ellipse be $a \cos \alpha$, $b \sin \alpha$; then will the equation to the tangent at that point be

$$\frac{\cos \alpha}{a} x + \frac{\sin \alpha}{b} y = 1.$$

Hence, by the conditions of the problem,

$$\tan \theta = \frac{b}{a} \tan \alpha,$$

$$\text{and } \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = -\frac{b}{a} \cot \alpha.$$

Hence, eliminating α ,

$$\frac{\tan^2 \theta + \tan \theta \tan \phi}{1 - \tan \theta \tan \phi} = -\frac{b^2}{a^2},$$

$$\left(1 - \frac{b^2}{a^2}\right) \tan \theta \tan \phi = -\left(\frac{b^2}{a^2} + \tan^2 \theta\right);$$

$$\begin{aligned} \therefore \tan \phi &= -\frac{b^2 \cos^2 \theta + a \sin^2 \theta}{(a^2 - b^2) \sin \theta \cos \theta} \\ &= \frac{(a^2 + b^2)(\cos^2 \theta + \sin^2 \theta) - (a^2 - b^2)(\cos^2 \theta - \sin^2 \theta)}{2(a^2 - b^2) \sin \theta \cos \theta} \\ &= \frac{a^2 + b^2 - (a^2 - b^2) \cos 2\theta}{(a^2 - b^2) \sin 2\theta}; \end{aligned}$$

$$\therefore (a^2 - b^2) (\sin 2\theta \tan \phi + \cos 2\theta) = a^2 + b^2,$$

$$\text{and } (a^2 - b^2) \cos(2\theta - \phi) = (a^2 + b^2) \cos \phi.$$

(2). Again, since

$$\tan \phi = -\frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{(a^2 - b^2) \sin \theta \cos \theta},$$

$$\begin{aligned}
\therefore - \int_0^\pi \phi d\theta &= \int_0^\pi \tan^{-1} \frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{(a^2 - b^2) \sin \theta \cos \theta} d\theta \dots \dots \dots (1) \\
&= \int_0^\pi \tan^{-1} \frac{b^2 \cos^2 (\pi - \theta) + a^2 \sin^2 (\pi - \theta)}{(a^2 - b^2) \sin \theta \cos \theta} d\theta \\
&= \int_0^\pi \left\{ m\pi - \tan^{-1} \frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{(a^2 - b^2) \sin \theta \cos \theta} \right\} d\theta \dots \dots (2) \\
&= \frac{1}{2} \int_0^\pi m\pi d\theta \text{ by adding (1) and (2)} \\
&= \frac{1}{2} (m\pi^2),
\end{aligned}$$

where m is a constant integer to be determined.

For this purpose we observe, first, that its value is independent of any relation between a and b ; and secondly, that if $a = b$ the ellipse becomes a circle and ϕ always $= \frac{1}{2}\pi$. Hence in this case

$$\int_0^\pi \phi d\theta = \int_0^\pi \frac{\pi}{2} d\theta = \frac{\pi^2}{2} \text{ and } m = -1.$$

Hence also, in all cases, $m = -1$ and

$$\int_0^\pi \phi d\theta = \frac{\pi^2}{2}.$$

3. Through a given point B (fig. 84) of the axis of x a line is drawn parallel to the axis of y : to any point Q of this line another straight line is drawn from the origin and produced to P until $PQ = BQ$. Find the equation to the locus of P , trace the curve, and find the whole area included between the curve and the asymptote.

Extend the geometrical description so as to include the whole of the curve given by the equation.

Let $AB = a$, $AP = r$, $PAB = \theta$. Then

$$QP = r - a \sec \theta,$$

$$QB = a \tan \theta;$$

$$\therefore r = a (\sec \theta + \tan \theta),$$

the polar equation to the curve.

Differentiating,

$$\frac{dr}{d\theta} = a \sec \theta (\tan \theta + \sec \theta)$$

$$= r \sec \theta,$$

$$\therefore r^2 \frac{d\theta}{dr} = r \cos \theta = a (1 + \sin \theta).$$

When $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, $r = \infty$ and $r^2 \frac{d\theta}{dr} = 2a$ and 0 ; hence the axis of y , and the line DCD parallel to it such that $AC = 2a$, are asymptotes to the curve:

$$\theta = 0, \quad r = a, \quad \frac{1}{r} \frac{dr}{d\theta} = 1,$$

$$\theta > 0, \quad < \frac{1}{2}\pi, \quad r \text{ is positive};$$

$$\theta > \frac{\pi}{2} < \pi, \quad r = -a \frac{1 + \sin \theta}{\cos \theta} \text{ is negative};$$

$$\theta = \pi, \quad r = -a, \quad \frac{1}{r} \frac{dr}{d\theta} = -1,$$

$$\theta > \pi < \frac{3\pi}{2}, \quad r = -a \frac{1 - \sin \theta}{\cos \theta} \text{ is negative};$$

$$\theta = \frac{1}{2}3\pi, \quad r = \infty,$$

$$\theta > \frac{3\pi}{2} < 2\pi, \quad r = a \frac{1 - \sin \theta}{\cos \theta} \text{ is positive};$$

the negative values of θ give no new branch of the curve.

Hence the curve is of the form represented in (fig. 85).

To find the area (A) included between the curve and the asymptote DCD' . Produce AP to meet the asymptote in R ; then the element of the area A

$$\begin{aligned} \delta A &= \frac{1}{2}(AR^2 - AP^2) \quad \delta A = \frac{1}{2}\{(2a \sec \theta)^2 - a^2 (\sec \theta + \tan \theta)^2\} \delta \theta \\ &= \frac{1}{2}a^2 (3 \sec^2 \theta - 2 \sec \theta \tan \theta - \tan^2 \theta) \delta \theta \\ &= \frac{1}{2}a^2 (2 \sec^2 \theta - 2 \sec \theta \tan \theta + 1) \delta \theta; \end{aligned}$$

$$\begin{aligned} \therefore A &= \frac{1}{2}a^2 \left(2 \tan \theta - \frac{2}{\cos \theta} + \theta \right) + C \\ &= \frac{1}{2}a^2 \left\{ \theta - 2 \left(\frac{1 - \sin \theta}{1 + \sin \theta} \right)^{\frac{1}{2}} \right\}; \end{aligned}$$

from $\theta = 0$ to $\theta = \frac{1}{2}\pi$ gives $\frac{1}{2}A$,

$$\therefore A = (\tfrac{1}{2}\pi + 2) a^2,$$

the required area.

If P' be the point where AP cuts the branch BP' of the curve, it is evident from the tracing of the curve that $QP' = QB$: hence the curve may be described as the locus of the point P on the line AR whose distance from Q equals QB .

GEOMETRY OF THREE DIMENSIONS.

1848.

1. IF three chords be drawn mutually at right angles through a fixed point within a surface of the second order whose equation is $u = 0$, shew that $\Sigma \frac{1}{Rr}$ will be constant, where R and r are the two portions into which any one of the chords drawn through the fixed point is divided by that point.

Prove also that the same will be true, if instead of the fixed point there be substituted any point in the surface whose equation is $u = c$.

We shall prove the second part of this only, since it manifestly includes the first.

Let the equation to the surface $u = 0$, referred to its centre and axes, be

$$Ax^2 + By^2 + Cz^2 = 1 \dots\dots\dots (1).$$

Let α, β, γ be the point through which the lines are drawn; then, since it always lies on the surface $u = c$, we have

$$A\alpha^2 + B\beta^2 + C\gamma^2 = 1 + c \dots\dots\dots (2).$$

Let $l_1 m_1 n_1, l_2 m_2 n_2, l_3 m_3 n_3$, be the direction-cosines of the lines, then their equations are

$$\frac{x - \alpha}{l_1} = \frac{y - \beta}{m_1} = \frac{z - \gamma}{n_1} = \rho_1 \text{ say} \dots\dots\dots (3),$$

$$\frac{x - \alpha}{l_2} = \frac{y - \beta}{m_2} = \frac{z - \gamma}{n_2} = \rho_2 \dots\dots\dots (4),$$

$$\frac{x - \alpha}{l_3} = \frac{y - \beta}{m_3} = \frac{z - \gamma}{n_3} = \rho_3 \dots\dots\dots (5),$$

$(l_1 m_1 n_1)$, $(l_2 m_2 n_2)$, $(l_3 m_3 n_3)$ being, since (3), (4), (5) are at right angles to one another, subject to the conditions

$$l_1^2 + l_2^2 + l_3^2 = 1 \dots \dots \dots (6),$$

$$m_1^2 + m_2^2 + m_3^2 = 1 \dots \dots \dots (7),$$

$$n_1^2 + n_2^2 + n_3^2 = 1 \dots \dots \dots (8).$$

Where (3) meets (1) we have, substituting for xyz in terms of ρ_1 ,

$$A(\alpha + l_1 \rho_1)^2 + B(\beta + m_1 \rho_1)^2 + C(\gamma + n_1 \rho_1)^2 = 1.$$

The roots of this, considered as an equation in ρ_1 , are $R_1 r$; hence

$$\begin{aligned} \frac{1}{Rr} &= \frac{Al_1^2 + Bm_1^2 + Cn_1^2}{A\alpha^2 + B\beta^2 + C\gamma^2 - 1} \\ &= \frac{Al_1^2 + Bm_1^2 + Cn_1^2}{c}. \end{aligned}$$

Similar expressions resulting from (4) and (5), we get by (6), (7), (8),

$$\Sigma \frac{1}{Rr} = \frac{A + B + C}{c},$$

which is constant.

2. Find the locus of the foot of the perpendicular let fall from the origin on the tangent plane to the surface $xyz = a^3$; point out the general form of the required surface, and find the whole included volume.

The equation to the tangent plane to the given surface at a point (xyz) , is

$$\frac{x_1}{x} + \frac{y_1}{y} + \frac{z_1}{z} = 3.$$

The equations to the perpendicular on this plane from the origin, are

$$xx_1 = yy_1 = zz_1.$$

At the intersection of these we have

$$xx_1 = yy_1 = zz_1 = \frac{x_1^2 + y_1^2 + z_1^2}{3};$$

therefore, since $xyz = a^3$, we get as the equation to the locus required,

$$(x_1^2 + y_1^2 + z_1^2)^3 = 27a^3 x_1 y_1 z_1.$$

The form of this surface will be that of four similar symmetrical pear-shaped portions, meeting in a point at the origin, and lying in the octants $+++$, $+-$, $-+-$, $---$.

The equation to the surface, transformed to polar coordinates, becomes

$$r^3 = 27a^3 \cos \theta \sin^2 \theta \cos \phi \sin \phi.$$

And if V' be the volume of one of the portions

$$V' = \frac{1}{2} \iiint r^2 \sin \theta \, dr \, d\theta \, d\phi,$$

between proper limits,

$$\begin{aligned} &= \frac{9}{2} a^3 \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \cos \theta \sin^3 \theta \cos \phi \sin \phi \, d\theta \, d\phi \\ &= \frac{9}{8} a^3 \int_0^{2\pi} \cos \phi \sin \phi \, d\phi \\ &= \frac{9}{16} a^3; \end{aligned}$$

therefore if V be the whole volume of the surface,

$$V = 4 V' = \frac{9}{4} a^3.$$

3. A plane moves so as always to enclose between itself and a given surface S a constant volume; prove that the envelope of the system of such planes is the same as the locus of the centres of gravity of the portions of the planes comprised within S .

Conceive the plane to receive a small twist about any straight line passing through the centre of gravity of the portion comprised within S ; then, whatever portion is cut off from the enclosed volume on one side of this line, an equal portion will be added to it on the other,* so that, by the conditions of the problem, the plane will pass from any one position to the consecutive one by turning about a line passing through the centre of gravity of the portion comprised within S . Therefore the envelope of the planes will be the same as the locus of the centres of gravity of the portions of the planes comprised within S .

* See *Cambridge and Dublin Mathematical Journal*, vol. iii. p. 181.

4. OA , OB , OC , are three straight lines mutually at right angles, and a luminous point is placed at C ; shew that when the quantity of light received upon the triangle AOB is constant, the curve which is always touched by AB will be an hyperbola whose equation referred to the axes OA , OB , is $(y - mx)(x - my) = mc^2$, where $OC = c$, and m is a constant quantity.

With C as centre, and CO as radius, describe a spherical surface, then the quantity of light received on the triangle AOB is the same as that received by the spherical triangle $CA'B'$ intercepted between the planes COA , COB , CBA , and will therefore be proportional to the area of that surface. But if S be this area,

$$\begin{aligned} S &= 2\pi r^2 (A'OB' + OA'B' + OB'A' - \pi) \\ &= 2\pi r^2 (OA'B' + OB'A' - \tfrac{1}{2}\pi), \end{aligned}$$

since AOB is a right angle.

Therefore if the quantity of light received by the triangle be constant, $OA'B' + OB'A'$ must be so, $= 2\alpha$ suppose.

Let the angle $OB'A' = \alpha + \theta$, then $OA'B'$ will $= \alpha - \theta$, and the equation to the plane ABC will be

$$\cos(\alpha + \theta)x + \cos(\alpha - \theta)y + \{1 - \cos^2(\alpha + \theta) - \cos^2(\alpha - \theta)\}^{\frac{1}{2}}z = p;$$

p will be determined from the consideration that where this meets the axis of z , we have $z = c$;

$$\therefore \{1 - \cos^2(\alpha + \theta) - \cos^2(\alpha - \theta)\}^{\frac{1}{2}} = c,$$

therefore the equation to AB is

$$\begin{aligned} \cos(\alpha + \theta)x + \cos(\alpha - \theta)y &= \{1 - \cos^2(\alpha + \theta) - \cos^2(\alpha - \theta)\}^{\frac{1}{2}}c \\ &= (-\cos 2\alpha \cos 2\theta)^{\frac{1}{2}}c \dots\dots\dots (1). \end{aligned}$$

The quantity $(-\cos 2\alpha \cos 2\theta)^{\frac{1}{2}}$ is real, since 2α is greater than a right angle and less than two right angles, and therefore $\cos 2\alpha$ negative.

Putting $\tan \alpha = n$, $\tan \theta = t$, (1) becomes

$$(1 - nt)x + (1 + nt)y = \{(n^2 - 1)(1 - t^2)\}^{\frac{1}{2}}c,$$

and we have to find the locus of ultimate intersections of this line, subject to the variation of t .

Clearing the equation of radicals and arranging according to powers of t , it becomes

$$t^2 \{n^2 (x-y)^2 + (n^2 - 1) c^2\} + 2tn (y^2 - x^2) + (x+y)^2 - (n^2 - 1)c^2 = 0.$$

Eliminating t between this equation and its derivative, we get

$$\{(x+y)^2 + (n^2 - 1) c^2\} \{n^2 (x-y)^2 + (n^2 - 1) c^2\} = n^2 (y^2 - x^2)^2,$$

which may be reduced to

$$(x+y)^2 - n^2 (x-y)^2 = (n^2 - 1) c^2,$$

$$\text{or } \{(1-n)x + (1+n)y\} \{(1+n)x + (1-n)y\} = (n^2 - 1) c^2;$$

therefore putting $\frac{n-1}{n+1} = m$,

$$(y - mx)(x - my) = mc^2$$

is the equation to the curve always touched by AB .

In a manner similar to this may be solved the following problem, set in 1851.

Let a spherical surface whose centre is the origin of coordinates meet two of the coordinate planes in the great circles Zx , Zy ; also let the points P , Q be taken in Zx , Zy respectively, so as to make the surface of the spherical triangle PZQ constant: shew that the curve which is always touched by the great circle PQ has for its equations $x^2 + y^2 + z^2 = a^2$, and $xy = \frac{1}{2}a^2 \sin E$, where E is the spherical excess of the triangle PZQ .

The geometrical conditions of this problem are the same as those of the foregoing. Writing z for c , we have as the equation to the surface always touched by the plane through the centre,

$$\begin{aligned} xy &= \frac{m}{1+m^2} (x^2 + y^2 + z^2) \\ &= \frac{1}{2} \frac{n^2 - 1}{n^2 + 1} (x^2 + y^2 + z^2) \\ &= - \frac{x^2 + y^2 + z^2}{2} \cos 2\alpha. \end{aligned}$$

But $\pi + E = A'OB' + OA'B' + OB'A'$ in the previous notation

$$= \frac{1}{2}\pi + 2\alpha;$$

$$\therefore 2\alpha = \frac{1}{2}\pi + E,$$

and our equation becomes

$$xy = \frac{1}{2}(x^2 + y^2 + z^2) \sin E.$$

But since the curve is traced on the sphere, we have $x^2 + y^2 + z^2 = a^2$, and we get as the equations to the curve,

$$x^2 + y^2 + z^2 = a^2,$$

$$xy = \frac{1}{2}a^2 \sin E,$$

the required equations.

5. If O be a given point in a surface of the second order, and OA , OB , OC , any three chords passing through O mutually at right angles, shew that the plane ABC will always pass through a fixed point.

Take O as origin, and the three lines OA , OB , OC , in any position as axes; let the equation to the surface be

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z = 0.$$

Then the length of OA will be the value of x , when $y = 0$, $z = 0$; hence

$$OA = -\frac{2A''}{A};$$

$$\text{similarly } OB = -\frac{2B''}{B}, \quad OC = -\frac{2C''}{C},$$

and the equation to ABC will be

$$\frac{Ax}{A''} + \frac{By}{B''} + \frac{Cz}{C''} + 2 = 0 \dots\dots\dots (1).$$

And the equations to the normal at O are

$$\frac{x}{2A''} = \frac{y}{2B''} = \frac{z}{2C''} = \frac{r}{2(A''^2 + B''^2 + C''^2)^{\frac{1}{2}}} \dots\dots\dots (2),$$

where r is the distance from the origin of the point (xyz) .

Where (1) and (2) intersect, we have, dividing each term of (1) by the corresponding member of (2),

$$A + B + C + \frac{2(A''^2 + B''^2 + C''^2)^{\frac{1}{2}}}{r} = 0.$$

Now we may establish one relation among the nine coefficients of the equation to the surface. Let then $(A''^2 + B''^2 + C''^2)$ be constant, then the above equation shews that r varies inversely as $A + B + C$.

But it is known that if the equation be transformed into the form

$$Px^2 + Qy^2 + Rz^2 + 2P'x + 2Q'y + 2R'z = 0,$$

the quantities P, Q, R , are the roots of the equation

$$(S-A)(S-B)(S-C) - A''(S-A) - B''(S-B) - C''(S-C) - 2A'B'C' = 0.$$

Hence, by the theory of equations,

$$A + B + C = P + Q + R, \text{ a constant.}$$

Hence r , the distance from O of the point in which the plane ABC intersects the normal at O , is constant, therefore the plane ABC always passes through a fixed point.

1849.

1. If planes be drawn through any two generating lines of an hyperboloid which intersect, shew that they will cut the surface in another pair of generating lines.

Let the equation to the hyperboloid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \dots\dots\dots (1).$$

Now a plane, drawn through two intersecting generating lines of an hyperboloid, touches the hyperboloid at their point of intersection. Let then x', y', z' , be the coordinates of this point, then the equation to the plane will be

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} - \frac{z'z}{c^2} = 1 \dots\dots\dots (2),$$

x', y', z' , being subject to the condition

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - \frac{z'^2}{c^2} = 1 \dots\dots\dots (3),$$

(1) + 2 (2) + (3) gives

$$\left(\frac{x+x'}{a}\right)^2 + \left(\frac{y+y'}{b}\right)^2 - \left(\frac{z+z'}{c}\right)^2 = 4 \dots\dots\dots (4),$$

a condition which must be satisfied by the coordinates of any point where (2) intersects (1).

Now (4) may be put into the form

$$\left(\frac{x+x'}{a}\right)^2 - 4 = \left(\frac{z+z'}{c}\right)^2 - \left(\frac{y+y'}{b}\right)^2,$$

which may be written

$$\left(\frac{x+x'}{a} + 2\right) \left(\frac{x+x'}{a} - 2\right) = \left(\frac{z+z'}{c} + \frac{y+y'}{b}\right) \left(\frac{z+z'}{c} - \frac{y+y'}{b}\right),$$

shewing that where (2) meets (1) we have either

$$\frac{x+x'}{a} + 2 = k \left(\frac{z+z'}{c} + \frac{y+y'}{b}\right) \text{ and } \frac{x+x'}{a} - 2 = \frac{1}{k} \left(\frac{z+z'}{c} - \frac{y+y'}{b}\right),$$

representing one generating line, or

$$\frac{x+x'}{a} + 2 = k' \left(\frac{z+z'}{c} - \frac{y+y'}{b}\right) \text{ and } \frac{x+x'}{a} - 2 = \frac{1}{k'} \left(\frac{z+z'}{c} + \frac{y+y'}{b}\right)$$

representing another.

Hence if planes be drawn through any two generating lines of an hyperboloid which intersect, they will cut the surface in another pair of generating lines.

2. If $u = f(x, y, z)$ be a rational function of x, y, z , and if $u = 0$ be the equation to a surface, for a point (a, b, c) of which all the partial differential coefficients of u as far as those of the $(n-1)^{\text{th}}$ order vanish, shew that the conical surface whose equation is

$$\left\{ (x-a) \frac{d}{da} + (y-b) \frac{d}{db} + (z-c) \frac{d}{dc} \right\}^n f(a, b, c) = 0,$$

will touch the proposed surface at the point (a, b, c) .

Let
$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \dots\dots\dots (1)$$

be the equations to any line passing through (a, b, c) .

Denoting each member of (1) by r , we shall obtain the other points of intersection of (1) with $u = 0$ by writing

$$a + lr \text{ for } x, \quad b + mr \text{ for } y, \quad c + nr \text{ for } z$$

in the equation $u = 0$. This gives, developing by Taylor's Theorem,

$$\begin{aligned} f(a, b, c) + \left(l \frac{d}{da} + m \frac{d}{db} + n \frac{d}{dc} \right) f(a, b, c) r \\ + \frac{1}{1.2} \left(l \frac{d}{da} + m \frac{d}{db} + n \frac{d}{dc} \right)^2 f(a, b, c) r^2 + \dots = 0, \end{aligned}$$

which, since $\frac{d^m u}{da^m} = \frac{d^m u}{db^m} = \frac{d^m u}{dc^m} = 0$ for all values of m less than n

becomes, dividing out by $\frac{r}{1.2\dots n}$,

$$\begin{aligned} \left(l \frac{d}{da} + m \frac{d}{db} + n \frac{d}{dc} \right)^n f(a, b, c) \\ + \frac{1}{n+1} \left(l \frac{d}{da} + m \frac{d}{db} + n \frac{d}{dc} \right)^{n+1} f(a, b, c) r + \dots = 0 \dots (2). \end{aligned}$$

If the line (1) touch the surface $u = 0$ at the point (a, b, c) equation (2) must be satisfied by making r indefinitely small; (2) will then become

$$\left(l \frac{d}{da} + m \frac{d}{db} + n \frac{d}{dc} \right)^n f(a, b, c) = 0,$$

a condition to be satisfied by the direction-cosines of (1) in order that it may touch $u = 0$ at the point (a, b, c) . To obtain the locus of all such lines, we must eliminate l, m, n from the above equation by means of (1). This gives

$$\left\{ (x-a) \frac{d}{da} + (y-b) \frac{d}{db} + (z-c) \frac{d}{dc} \right\}^n f(a, b, c) = 0$$

as the equation to the conical surface which touches $u = 0$ at the point (a, b, c) .

3. A rod AB is fixed to a universal joint to A , and another rod BP is connected to it by a universal joint at B : all directions of the rod being equally probable, find the chance of P lying between two spherical surfaces of given radii, whose common centre is A ; and shew that the chance of P lying within a given elementary portion of space containing the point P , varies inversely as AP_1 .

Let $AB = a$, $BP = b$.

The chance that P will lie between two spherical surfaces of given radii, is the chance that the angle ABP will lie between two values θ_1 and θ_2 , which correspond to the values r_1 and r_2 of AP , r_1 and r_2 being the radii of the spherical shell.

Now the chance that ABP will lie between θ and $\theta + \delta\theta$
 = area of zone described by P about B fixed, while θ has all values from θ to $\theta + \delta\theta$ \div surface of sphere generated by P about B fixed,

$$= \frac{2\pi b \sin \theta \times b \sin \theta}{4\pi b^2} = \frac{1}{2} \sin \theta d\theta.$$

Hence the chance required

$$\begin{aligned} &= \frac{1}{2} \int_{\theta_1}^{\theta_2} \sin \theta = \frac{1}{2} (\cos \theta_1 - \cos \theta_2) = \frac{1}{2} \left(\frac{a^2 + b^2 - r_1^2}{2ab} - \frac{a^2 + b^2 - r_2^2}{2ab} \right) \\ &= \frac{r_2^2 - r_1^2}{4ab}. \end{aligned}$$

The chance that P will lie in an element V of space about P_1
 = chance of falling in a spherical shell about A as centre of
 thickness δr , radius $r \times \frac{\text{volume of element}}{\text{volume of shell}}$

$$= \frac{2r\delta r}{4ab} \times \frac{V}{4\pi r^2 \delta r} = \frac{V}{8\pi ab} \cdot \frac{1}{r} \propto \frac{1}{r}.$$

4. Determine the condition to which the vertices of a system of cones which envelope an ellipsoid must be subject, in order that the centres of the ellipses of contact may be equidistant from the centre of the ellipsoid.

Let ξ, η, ζ , be the coordinates of the vertex of any one of the cones; then if the ellipse be referred to its centre and axes, the equation to the plane of contact will be

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 1 \dots\dots\dots (1).$$

The centre of the ellipse of contact will be the intersection of (1) with the straight line joining the centre of the ellipse with the vertex of the cone; its equations are

$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{z}{\zeta}.$$

Hence if h, k, l , be the coordinates of the centre of the ellipse,

$$h = \xi \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} \right)^{-1}, \quad k = \eta \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} \right)^{-1}, \quad l = \zeta \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} \right)^{-1}.$$

In order that the centres of the ellipses may be equidistant from the centre of the ellipsoid, we must have

$$h^2 + k^2 + l^2 = \text{constant}, \quad p^2 \text{ suppose;}$$

$$\therefore \xi^2 + \eta^2 + \zeta^2 = p^2 \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} \right),$$

the equation to the locus of the vertices.

5. Determine the form of the termination of a honeycomb cell on this principle, that if a sphere which will just pass through the hexagonal transverse section be dropped into the cell, the unoccupied space at the extremity of the cell shall be the least possible.*

Let abc (fig. 86) represent half of one of the rhomboidal plates, three of which close each hexagonal cell. ABC represent the eighth part of a sphere. Then, by the general principle of Envelopes (see *Cambridge and Dublin Mathematical Journal*, vol. iii. p. 181), the volume in question is least when the point of contact d is the centre of the rhomboid: or we must have $ac = 2ad$.

$$\text{Let } OA = a, \quad OD = r; \quad \therefore a = r \cos 30^\circ = r \frac{3^{\frac{1}{2}}}{2}.$$

* For this solution we are indebted to Mr. Goodwin.

$$\text{Let } Oac = \phi; \therefore a = Od = ad \tan \phi = \frac{1}{2} ac \tan \phi \\ = \frac{1}{2} r \operatorname{cosec} \phi \tan \phi;$$

$$\therefore r \frac{3^{\frac{1}{2}}}{2} = \frac{1}{2} r \frac{1}{\cos \phi};$$

$$\therefore \cos \phi = \frac{1}{3^{\frac{1}{2}}}.$$

$$\text{Again, } bd = ED \sin 60 = r \frac{3^{\frac{1}{2}}}{2}, \text{ and } ad = \frac{1}{2} r \operatorname{cosec} \phi;$$

$$\therefore \tan bad = \frac{bd}{ad} = 3^{\frac{1}{2}} \sin \phi = 3^{\frac{1}{2}} \left(1 - \frac{1}{3}\right)^{\frac{1}{2}} = 2^{\frac{1}{2}}.$$

These are the angles required.

6. The tangent plane to a surface S cuts an ellipsoid, and the locus of the vertex of the cone which touches the ellipsoid in the curve of intersection is another surface S' . Prove that S and S' are reciprocal, that is, that S may be generated from S' in the same manner as S' has been generated from S .

Take the axes of the ellipsoid as coordinate axes, and let the equation to S be

$$S = 0.$$

That to its tangent plane at any point xyz , is

$$(x_1 - x) \frac{dS}{dx} + (y_1 - y) \frac{dS}{dy} + (z_1 - z) \frac{dS}{dz} = 0.$$

This may be put under the form

$$\frac{x_1 \frac{dS}{dx} + y_1 \frac{dS}{dy} + z_1 \frac{dS}{dz}}{x \frac{dS}{dx} + y \frac{dS}{dy} + z \frac{dS}{dz}} = 1.$$

If ξ, η, ζ , be the coordinates of the vertex of the cone touching the ellipsoid in the curve of intersection with this plane, we have

$$\frac{\xi}{a^2} = \frac{\frac{dS}{dx}}{x \frac{dS}{dx} + y \frac{dS}{dy} + z \frac{dS}{dz}},$$

with similar expressions for η and ζ .

Again, if with (xyz) a point of S as vertex we describe a cone touching the ellipsoid, the equation to the plane of contact will be

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} + \frac{z\zeta}{c^2} = 1,$$

ξ, η, ζ being its current coordinates. To find the locus of ultimate intersection of these planes, eliminate x, y, z between the differential of the preceding equation, and of

$$S = 0;$$

$$\text{this gives } \lambda \frac{\xi}{a^2} + \frac{dS}{dx} = 0,$$

$$\lambda \frac{\eta}{b^2} + \frac{dS}{dy} = 0,$$

$$\lambda \frac{\zeta}{c^2} + \frac{dS}{dz} = 0.$$

Multiplying these equations in order by x, y, z , and adding, we get

$$\lambda + x \frac{dS}{dx} + y \frac{dS}{dy} + z \frac{dS}{dz} = 0;$$

$$\therefore \frac{\xi}{a^2} = \frac{\frac{dS}{dx}}{x \frac{dS}{dx} + y \frac{dS}{dy} + z \frac{dS}{dz}},$$

with similar expressions for η, ζ .

Hence the locus of ξ, η, ζ is S' .

That is, the locus of the vertex of the cone touching the ellipsoid in its curve of intersection with a tangent plane to S is the same as the envelope of the plane of contact when a cone is drawn from a point of S as vertex, circumscribing the ellipsoid. This holds for all surfaces, therefore for S' .

But from the mode of generation of S' , it is easy to see that the envelope of the planes of contact of cones drawn from its points as vertices is S ; therefore, by what has been proved, the locus of the vertices of the cones touching the ellipsoid in its curves of intersection with the tangent planes to S' is S , that

is, S may be generated from S' in the same manner as S' was from S , or S and S' are reciprocal.

1850.

1. If from a point O be drawn any two lines to the polar plane of O in a surface of the second order and meet the plane in A and B , and if the central conjugate plane of OA meet OB in C , and the central conjugate plane of OB meet OA in D , CD is parallel to AB .

Take that diameter of the ellipsoid which passes through O , and two diameters conjugate to it, as axes. Let the equation to the ellipsoid be

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1.$$

Let ξ be the distance of O from the centre, then the equation to its polar plane is

$$x = \frac{a'^2}{\xi}.$$

Let the coordinates of A be $\frac{a'^2}{\xi}, y_1, z_1$; of B , $\frac{a'^2}{\xi}, y_2, z_2$. Then the equations to AB are

$$x = \frac{a'^2}{\xi}, \quad \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2} \dots \dots \dots (1).$$

The equations to OA will be

$$\frac{x - \frac{a'^2}{\xi}}{\frac{a'^2}{\xi} - \xi} = \frac{y}{y_1} = \frac{z}{z_1} = r_1 \text{ suppose } \dots \dots \dots (2),$$

therefore the equation to its central conjugate plane is

$$\left(\frac{a'^2}{\xi} - \xi \right) \frac{x}{a'^2} + \frac{y_1 y}{b'^2} + \frac{z_1 z}{c'^2} = 0 \dots \dots \dots (3).$$

Similarly, the equations to OB are

$$\frac{x - \frac{a'^2}{\xi}}{\frac{a'^2}{\xi} - \xi} = \frac{y}{y_2} = \frac{z}{z_2} = r_2 \dots \dots \dots (4),$$

therefore that to its central conjugate plane is

$$\left(\frac{a'^2}{\xi} - \xi\right) \frac{x}{a'^2} + \frac{y_2 y}{b'^2} + \frac{z_2 z}{c'^2} = 0 \dots\dots\dots (5).$$

At the point C , the intersection of (3) with (4), we have

$$r_2 = \frac{1 - \frac{\xi^2}{a'^2}}{\left(\frac{a'^2}{\xi} - \xi\right)^2 \frac{1}{a'^2} + \frac{y_1 y_2}{b'^2} + \frac{z_1 z_2}{c'^2}}.$$

Similarly, it will be seen that at D , the intersection of (2) and (5), we have

$$r_1 = \frac{1 - \frac{\xi^2}{a'^2}}{\left(\frac{a'^2}{\xi} - \xi\right)^2 \frac{1}{a'^2} + \frac{y_1 y_2}{b'^2} + \frac{z_1 z_2}{c'^2}} \\ = r_2,$$

the values of x at C and D are each equal to $\frac{a'^2}{\xi} + \left(\frac{a'^2}{\xi} - \xi\right) r_1$, therefore the equations to CD are

$$x = \frac{a'^2}{\xi} + \left(\frac{a'^2}{\xi} - \xi\right) r_1, \quad \frac{y - y_1 r_1}{y_1 - y_2} = \frac{z - z_1 r_1}{z_1 - z_2},$$

by comparing which equations with (1) we see that CD is parallel to AB .

2. A plane moves so as always to cut off from an ellipsoid the same volume; shew that it will in every position touch a similar and concentric ellipsoid.

If a plane be drawn touching the interior of two similar and concentric ellipsoids, the point of contact will be the centre of its elliptic section made by the exterior one. Now conceive this plane to receive a small twist about any diameter: it will still remain in contact with the interior ellipsoid, and whatever portion is taken from the volume intercepted between it and the exterior ellipsoid on one side, will be added to it on the other, therefore that volume will be unaltered. Hence conversely, it follows that if a plane move so as always to cut off

from an ellipsoid the same volume, the surface which it always touches will be a similar and concentric ellipsoid.

3. If $F(x, y, c) = 0$ be the equation of a system of curves, where c is a variable parameter, and $\phi(x, y) = 0$ the equation of the envelope of the system; shew that $\phi(x, y) = 0$ is the equation of a cylinder whose intersection with the surface $F(x, y, z) = 0$ is the locus of points which in sections parallel to the planes of yx, zx , have their tangents parallel to the axis of z .

Ex. The cone whose equation is $x^2 + y^2 + z^2 = (lx + my + nz)^2$ is cut by planes parallel to the planes of yz and zx ; find the loci of the extremities of the diameters of the sections which are conjugate to the vertical diameter.

(α). The equation $\phi(x, y) = 0$ (1)

results from the elimination of c between the equations

$$F(x, y, c) = 0,$$

$$\text{and } \frac{dF}{dc} = 0.$$

It will therefore be also obtained by eliminating z between

$$F(x, y, z) = 0 \dots\dots\dots (2),$$

$$\text{and } \frac{dF}{dz} = 0.$$

Hence, where the surfaces represented by (1) and (2) intersect, we have

$$\frac{dF}{dz} = 0 \dots\dots\dots (3).$$

Now the equation to a tangent plane to (1), parallel to the plane of yz , is

$$(y_1 - y) \frac{dF}{dy} + (z_1 - z) \frac{dF}{dz} = 0.$$

If therefore $\frac{dF}{dz} = 0$, this becomes

$$y_1 = y,$$

which is evidently parallel to the axis of z .

Hence at the curve of intersection of the cylinder (1) and the surface (2), the tangent in a section parallel to the plane of yz is parallel to the axis of z .

Similarly it may be shewn that the tangent in a section parallel to the plane of xz is parallel to the axis of z .

(β). In the example, the tangent at the required points are parallel to the vertical diameter, that is to the axis of z , hence we get the locus required by eliminating z between

$$F(x, y, z) = x^2 + y^2 + z^2 - (lx + my + nz)^2 = 0,$$

$$\text{and } \frac{dF}{dz} = 2z - 2n(lx + my + nz) = 0.$$

The latter equation gives

$$z = n \frac{lx + my}{1 - n^2}.$$

$$\begin{aligned} \text{Hence } x^2 + y^2 &= (lx + my)^2 \left(1 + \frac{n^2}{1 - n^2} \right)^2 - n^2 \frac{(lx + my)^2}{(1 - n^2)^2} \\ &= \frac{(lx + my)^2}{1 - n^2}, \end{aligned}$$

is the equation to a cylinder, whose intersection with the given surface is the required locus.

4. If x, y, z , be the coordinates of any point P on the surface $f(x, y, z) = 0$, x', y', z' of a point P' on the surface $f(x', y', z') = 0$, and for any position of P , $x' = lx$, $y' = my$, $z' = nz$; and if the surfaces be such that when we take any two points P, Q on the first and two corresponding points P', Q' on the second, PQ' is equal to $P'Q$; find the form of the surfaces.

Let ξ, η, ζ be the coordinates of Q .

Then $l\xi, m\eta, n\zeta$ are those of Q' ,

$$\therefore PQ'^2 = (x - l\xi)^2 + (y - m\eta)^2 + (z - n\zeta)^2,$$

$$P'Q^2 = (\xi - lx)^2 + (\eta - my)^2 + (\zeta - nz)^2,$$

and these are equal; hence we get

$(x - l\xi)^2 + (y - m\eta)^2 + (z - n\zeta)^2 = (\xi - lx)^2 + (\eta - my)^2 + (\zeta - nz)^2$,
 or $(1-l^2)x^2 + (1-m^2)y^2 + (1-n^2)z^2 = (1-l^2)\xi^2 + (1-m^2)\eta^2 + (1-n^2)\zeta^2$,
 for all values of x, y, z ; ξ, η, ζ , consistent with the equation to the surface $f(x, y, z) = 0$.

We must therefore have

$(1-l^2)x^2 + (1-m^2)y^2 + (1-n^2)z^2 = \text{constant}$, a^2 suppose,
 which determines the form of the required surfaces, which are evidently central surfaces of the second order, of which the axes of coordinates are principal axes.

5. It is not possible to fill any given space with a number of regular polyhedrons of the same kind except cubes, but this may be done by means of tetrahedrons and octahedrons which have equal faces, by using twice as many of the former as of the latter.

Consider two octahedra so placed that two of their edges shall coincide, and the squares of which they are sides be in the same plane. Let AB (fig. 87) be either of these edges, C a vertex of one octahedron, not lying in the plane of the squares, D the corresponding vertex of the other. Then $CD =$ a side of the square $= AB = CA = CB = BD = AD$, by definition of a regular octahedron. Hence $CADB$ is a regular tetrahedron. Hence if we have a number of octahedra, so placed that one plane shall contain a square section of each, and each edge of each such section coincide with one edge of each of the adjacent sections, an equal number of tetrahedra will fill up the vacant space above the plane, and therefore by using twice as many tetrahedra as octahedra, we fill up the space above and below.

6. Prove that the tangent plane at any point of the surface

$$(ax)^2 + (by)^2 + (cz)^2 = 2(bcyz + cazx + abxy),$$

intersects the surface $ayz + bzx + cxy = 0$ in two straight lines at right angles to one another.

The equation

$$(ax)^2 + (by)^2 + (cz)^2 = 2(bcyz + cazx + abxy)$$

may be put into the form

$$(ax)^{\frac{1}{2}} + (by)^{\frac{1}{2}} + (cz)^{\frac{1}{2}} = 0 \dots\dots\dots(1).$$

Also $ayz + bzx + cxy = 0$ may be written

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0 \dots\dots\dots(2).$$

The equation to the tangent plane to (1) at any point (xyz) is

$$\left(\frac{a}{x}\right)^{\frac{1}{2}} x_1 + \left(\frac{b}{y}\right)^{\frac{1}{2}} y_1 + \left(\frac{c}{z}\right)^{\frac{1}{2}} z_1 = 0 \dots\dots\dots(3).$$

Let $(l_1 m_1 n_1)$, $(l_2 m_2 n_2)$ be the direction-cosines of the lines in which (2) meets (3), then the condition of these being at right angles to one another, is

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \dots\dots\dots(4).$$

Now where (2) meets (3), we have, writing $x_1 y_1 z_1$ for xyz in (2),

$$a \left(\frac{a}{x}\right)^{\frac{1}{2}} = b \left(\frac{b}{y}\right)^{\frac{1}{2}} + c \left(\frac{c}{z}\right)^{\frac{1}{2}} + (bc)^{\frac{1}{2}} \left\{ \left(\frac{c}{y}\right)^{\frac{1}{2}} \frac{y_1}{z_1} + \left(\frac{b}{z}\right)^{\frac{1}{2}} \frac{z_1}{y_1} \right\},$$

a quadratic in $\frac{y_1}{z_1}$ whose roots are $\frac{m_1}{n_1}$, $\frac{m_2}{n_2}$. Hence

$$\frac{m_1 m_2}{n_1 n_2} = \left(\frac{by}{cz}\right)^{\frac{1}{2}};$$

$$\text{similarly } \frac{l_1 l_2}{n_1 n_2} = \left(\frac{ax}{cz}\right)^{\frac{1}{2}};$$

$$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 \propto (ax)^{\frac{1}{2}} + (by)^{\frac{1}{2}} + (cz)^{\frac{1}{2}} = 0 \text{ by (1).}$$

Hence the tangent plane at any point of (1) cuts (3) in two straight lines at right angles to one another.

7. A certain territory is bounded by two meridian circles, and by two parallels of latitude which differ in longitude and latitude respectively by one degree, and is known to lie within certain limits of latitude: find the probable superficial area.

First, to find the chance that the centre of the territory which lies between known limits α, β of latitude lies in the zone between parallels of latitude l and $l + \delta l$.

$$\begin{aligned} \text{This chance} &= \frac{\text{area of zone breadth } \delta l}{\text{area of zone breadth } (\alpha - \beta)}, \\ &= \frac{2\pi r \cos l \, r \delta l}{\int_{\beta}^{\alpha} 2\pi r \cos l \, r \delta l} \quad (r \text{ the radius of the earth,}) \\ &= \frac{\cos l \, \delta l}{\sin \alpha - \sin \beta}. \end{aligned}$$

Then the probable superficies of the territory

$$= \int_{\beta}^{\alpha} A \frac{\cos l \, \delta l}{\sin \alpha - \sin \beta},$$

A being the area of the territory when its centre lies in the zone between the parallels l and $l + \delta l$;

$$\begin{aligned} \therefore A &= \frac{1}{360} \int_{l-30'}^{l+30'} 2\pi r \cos \phi \, r \delta \phi, \\ &= \frac{\pi r^2}{180} \{\sin(l + 30') - \sin(l - 30')\}, \\ &= \frac{2\pi r^2 \sin 30'}{180} \cos l. \end{aligned}$$

Therefore the probable superficies,

$$\begin{aligned} &= \int_{\beta}^{\alpha} \frac{2\pi r^2 \sin 30'}{180(\sin \alpha - \sin \beta)} \cos^2 l \, \delta l, \\ &= \frac{\pi r^2 \sin 30'}{180(\sin \alpha - \sin \beta)} \left\{ \alpha - \beta + \frac{1}{2}(\sin 2\alpha - \sin 2\beta) \right\}. \end{aligned}$$

1851.

1. A line passing through a fixed point and having the sum of its inclinations to two fixed lines through the same point constant, generates a cone of the second order.

Any section perpendicular to either of the fixed lines has for a focus its intersection with the fixed line.

(α). Let OP be the moving line, OS , OH the fixed lines. Take the lines bisecting the angle SOH and its interior angle respectively, as axes of x and y . Describe a spherical surface about O , cutting OP , OS , OH in P , S , H ; join SH , SP , HP by arcs of great circles, then by the conditions of the problem

$$SP + HP = \text{constant, } 2\alpha \text{ suppose.}$$

Bisect SH in X , join OX , let $SX = HX = \beta$. Draw PM an arc of a great circle, perpendicular to SH , let $XM = \theta$, $PM = \phi$. Then, by Napier's rules

$$\begin{aligned}\cos SP &= \cos SM \cdot \cos MP, \\ &= \cos(\beta + \theta) \cos \phi, \\ \cos HP &= \cos HM \cdot \cos MP, \\ &= \cos(\beta - \theta) \cos \phi.\end{aligned}$$

$$\text{Now } \cos SP + \cos HP = 2 \cos \frac{SP + HP}{2} \cos \frac{SP - HP}{2};$$

$$\therefore \cos \beta \cos \theta \cos \phi = \cos \alpha \cos \frac{SP - HP}{2},$$

$$\text{and } \cos HP - \cos SP = 2 \sin \frac{SP + HP}{2} \sin \frac{SP - HP}{2};$$

$$\therefore \sin \beta \sin \theta \cos \phi = \sin \alpha \sin \frac{SP - HP}{2},$$

therefore adding squares

$$\cos^2 \phi \left(\frac{\cos^2 \beta \cos^2 \theta}{\cos^2 \alpha} + \frac{\sin^2 \beta \sin^2 \theta}{\sin^2 \alpha} \right) = 1 \dots \dots \dots (1).$$

$$\text{Now } \sin \phi = \sin PM = \frac{z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}},$$

$$\therefore \cos^2 \phi = \frac{x^2 + y^2}{x^2 + y^2 + z^2},$$

$$\cos \theta = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}, \quad \sin \theta = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}};$$

therefore equation (1) becomes

$$x^2 \frac{\cos^2 \beta}{\cos^2 \alpha} + y^2 \frac{\sin^2 \beta}{\sin^2 \alpha} = x^2 + y^2 + z^2 \dots \dots \dots (2),$$

shewing that the locus of P is a cone of the second order.

(β). Let $x \cos \beta + y \sin \beta = p$ (3),
be the equation of a plane perpendicular to OH .

Where this meets OH , we have

$$x = p \cos \beta, \quad y = p \sin \beta.$$

The distance of any point (xyz) in (3) from this point is

$$(x^2 + y^2 + z^2 - p^2)^{\frac{1}{2}}.$$

If the point (xyz) also lie in (2), this becomes

$$\left(\frac{x^2 \cos^2 \beta}{\cos^2 \alpha} + \frac{y^2 \sin^2 \beta}{\sin^2 \alpha} - p^2 \right)^{\frac{1}{2}},$$

or substituting for y from (3)

$$\left\{ \frac{x^2 \cos^2 \beta}{\cos^2 \alpha} + \frac{(p - x \cos \beta)^2}{\sin^2 \alpha} - p^2 \right\}^{\frac{1}{2}},$$

which is equal to

$$\left(\frac{x^2 \cos^2 \beta}{\cos^2 \alpha \sin^2 \alpha} - \frac{2px \cos \beta}{\sin^2 \alpha} + \frac{p^2 \cos^2 \alpha}{\sin^2 \alpha} \right)^{\frac{1}{2}},$$

$$\text{or } \frac{x \cos \beta}{\cos \alpha \sin \alpha} - p \frac{\cos \alpha}{\sin \alpha}.$$

Hence the distance of any point in the curve of intersection of (2) and (3) from the point of intersection of OH with (3) is a linear function of x , which is a property peculiar to the focus. Therefore any section perpendicular to either of the fixed lines has for a focus its intersection with the fixed line.*

2. The locus of the points in which a principal plane of a surface of the second order is intersected by the normals at the different points of a plane section of the surface is a conic section.

Let the equation to the surface referred to its principal planes, be

$$Ax^2 + By^2 + Cz^2 = 1 \dots \dots \dots (1),$$

and to the plane of section

$$lx + my + nz = p \dots \dots \dots (2).$$

* See Hearn on *Curves of the Second Order*, p. 65, et seqq.

Then those to the normal at (xyz) are

$$\frac{x_1 - x}{Ax} = \frac{y_1 - y}{By} = \frac{z_1 - z}{Cz}.$$

Where this meets the plane of yz , we have

$$y_1 = y \left(1 - \frac{B}{A}\right), \quad z_1 = z \left(1 - \frac{C}{A}\right),$$

$$\text{or } y = \frac{Ay_1}{A - B}, \quad z = \frac{Az_1}{A - C}.$$

Eliminating x between (1) and (2), and substituting the above values of y and z , we shall obtain the equation to the locus required, which may easily be seen to be of the second order.

3. Normals are drawn to a surface at points indefinitely near to and equidistant from a fixed point in the surface: determine and discuss the equation of the surface generated by the normals.

Take the fixed point as origin, and the principal planes through it as planes of yz and zx . Let the equation to the surface be

$$z = Ax^2 + By^2 + \dots$$

The equations to the normals at (x, y, z) , are

$$x_1 - x + \frac{dz}{dx} (z_1 - z) = 0, \quad y_1 - y + \frac{dz}{dy} (z_1 - z) = 0;$$

$$\text{or } x_1 - x + 2Ax(z_1 - Ax^2 - By^2 - \dots) = 0 \dots \dots (1),$$

$$y_1 - y + 2By(z_1 - Ax^2 - By^2 - \dots) = 0 \dots \dots (2).$$

Again, since the point (xyz) is always at the same distance from the origin, we have

$$x^2 + y^2 + z^2 = a^2,$$

$$\text{or } x^2 + y^2 + (Ax^2 + By^2 + \dots)^2 = a^2 \dots \dots (3).$$

The elimination of x, y between (1), (2), (3), would give the equation to the surface. But since the point (xyz) is always indefinitely near to the origin, x, y, z are always indefinitely

small, and we may neglect their powers higher than the second. Hence our equations become

$$x_1 - x + 2Az_1x = 0,$$

$$y_1 - y + 2Bz_1y = 0,$$

$$x^2 + y^2 = a^2.$$

Eliminating x, y between these, we get

$$\frac{x_1^2}{(2Az_1 - 1)^2} + \frac{y_1^2}{(2Bz_1 - 1)^2} = a^2,$$

the required equation to the surface.

This surface is evidently of the fourth order, and symmetrical with respect to the planes of yz and zx . Its section, by any plane parallel to the plane of xy is an ellipse, which becomes a circle when the distance z_1 of the cutting plane from that of $xy = \frac{1}{A+B}$. When $z_1 = \frac{1}{2A}$, the equation becomes $x_1 = 0$, shewing that the section is there a straight line parallel to the axis of y , and similarly when $z_1 = \frac{1}{2B}$, the section is a straight line parallel to the axis of x . The points where these lines meet the axis of z , are the centres of curvature of the principal sections for $\frac{1}{2A}, \frac{1}{2B}$ are the principal radii of curvature at the origin. When $z_1 > \frac{1}{2B}$ (supposing $A > B$), the area of the section continually increases, as manifestly ought to be the case, since the normals altogether diverge after $z_1 > \frac{1}{2B}$.

4. A plane is drawn through the axis of y , such that its trace upon the plane of zx touches the two circles in which the plane of zx meets the surface generated by the revolution round the axis of z of the circle $(x-a)^2 + z^2 = c^2 (c < a)$; find the equation to the curve of intersection of the plane and surface, and from this equation trace the curve.

The equation to the plane will be

$$\frac{z}{c} = \frac{x}{(a^2 - c^2)^{\frac{1}{2}}} \dots\dots\dots (1).$$

The equation to the surface, generated by the revolution round the axis of z of the given circle, is

$$\{(x^2 + y^2)^{\frac{1}{2}} - a\}^2 + z^2 = c^2,$$

which rationalized becomes

$$(x^2 + y^2 + z^2 + a^2 - c^2)^2 = 4a^2(x^2 + y^2) \dots\dots\dots (2).$$

To obtain the curve of intersection of (1) and (2), we must turn the planes of yz and xy round the axis of y till (1) coincides with the plane of xy , and then put $z = 0$. This is effected by writing

$$\frac{(a^2 - c^2)^{\frac{1}{2}}x - cz}{a} \text{ for } x,$$

$$\frac{cx + (a^2 - c^2)^{\frac{1}{2}}z}{a} \text{ for } z,$$

or since z is to be put $= 0$, $\frac{(a^2 - c^2)^{\frac{1}{2}}}{a}x$ for x , and $\frac{cx}{a}$ for z ; this reduces the equation to

$$(x^2 + y^2 + a^2 - c^2)^2 = 4\{(a^2 - c^2)x^2 + a^2y^2\},$$

$$\text{or } (x^2 + y^2 + a^2 - c^2)^2 = 4(a^2 - c^2)(x^2 + y^2) + 4c^2y^2,$$

which may be reduced to

$$x^2 + y^2 - a^2 + c^2 = \pm 2cy,$$

$$\text{or } x^2 + (y \pm c)^2 = a^2,$$

shewing that the curve is composed of two circles, the radius of each of which is a , and whose centres lie on the axis of y , on opposite sides of the origin, and at a distance from it $= c$.

5. Prove (one of) the two following properties:

(1). If (A) , (B) be two given spheres not intersecting each other, then every sphere which cuts (A) and (B) in given angles will touch two fixed spheres.

(2). If (A) , (B) be two given spheres cutting one another, then every sphere which cuts (A) and (B) in given angles will cut orthogonally a fixed sphere.

(1). Take the line joining the centres of the spheres for axis of x , and its middle point for origin: let a , $-a$ be the abscissæ of these centres, r_1 , r_2 the radii of (A) and (B) ; x , y , z the coordinates of the centre of a sphere which cuts (A) and (B) in given angles α , β ; r its radius. Then we must have

$$(x-a)^2 + y^2 + z^2 = r_1^2 + r^2 - 2r_1r \cos \alpha,$$

$$(x+a)^2 + y^2 + z^2 = r_2^2 + r^2 - 2r_2r \cos \beta.$$

Let $(b, 0, 0)$ be the coordinates of the centre of a sphere which this moveable sphere always touches.

Adding and subtracting the above equations, we find

$$x^2 + a^2 + y^2 + z^2 = \frac{1}{2}(r_1^2 + r_2^2) + r^2 - r(r_1 \cos \alpha + r_2 \cos \beta),$$

$$\text{and } -2xb = \frac{1}{2} \frac{b}{a} \{r_1^2 - r_2^2 - 2r(r_1 \cos \alpha - r_2 \cos \beta)\},$$

$$b^2 - a^2 = b^2 - a^2,$$

adding these three equations, we have

$$\begin{aligned} (x-b)^2 + y^2 + z^2 = r^2 - r \left\{ r_1 \cos \alpha + r_2 \cos \beta + \frac{b}{a} (r_1 \cos \alpha - r_2 \cos \beta) \right\}, \\ + \frac{1}{2} \left\{ r_1^2 + r_2^2 + \frac{b}{a} (r_1^2 - r_2^2) \right\} + b^2 - a^2 \dots\dots (1). \end{aligned}$$

It is evident that the moveable sphere will touch the sphere whose centre is at a distance from the origin if the right hand member of equation (1) be a perfect square, or if

$$\left\{ r_1 \cos \alpha + r_2 \cos \beta + \frac{b}{a} (r_1 \cos \alpha - r_2 \cos \beta) \right\}^2 = 2 \left\{ r_1^2 + r_2^2 + \frac{b}{a} (r_1^2 - r_2^2) \right\} + 4(b^2 - a^2),$$

a quadratic for the determination of b , shewing that there are two spheres which the moveable sphere always touches.

(2). Also it is evident that the moveable sphere will always cut orthogonally the sphere, the abscissa of whose centre is b , if the right-hand member of equation (1) assume the form

$$r^2 + c^2.$$

This it will do if

$$r_1 \cos \alpha + r_2 \cos \beta + \frac{b}{a} (r_1 \cos \alpha - r_2 \cos \beta) = 0;$$

$$\text{or } \frac{b}{a} = \frac{r_2 \cos \beta - r_1 \cos \alpha}{r_2 \cos \beta + r_1 \cos \alpha},$$

which determines the centre of the sphere.

DIFFERENTIAL EQUATIONS.

1848.

Assuming that $\sin x + \frac{\cos x}{x}$ is a particular integral of the equation

$$\frac{d^2 y}{dx^2} + \left(1 - \frac{2}{x^2}\right)y = 0 \dots\dots\dots(1),$$

find the complete integral of the equation

$$\frac{d^2 y}{dx^2} + \left(1 - \frac{2}{x^2}\right)y = x^2 \dots\dots\dots(2).$$

We see by substitution that not only

$$y = \sin x + \frac{\cos x}{x},$$

is a particular integral of equation (1), but also

$$y = \cos x - \frac{\sin x}{x}.$$

Hence the complete solution of (2) is

$$y = A \left(\sin x + \frac{\cos x}{x} \right) + B \left(\cos x - \frac{\sin x}{x} \right),$$

where A and B are arbitrary constants.

Now assume as the integral of equation (2),

$$y = A \left(\sin x + \frac{\cos x}{x} \right) + B \left(\cos x - \frac{\sin x}{x} \right),$$

where A and B are now functions of x which have to be determined.

By the usual assumptions of the method of variable parameters, we find

$$\frac{dy}{dx} = A \left(\cos x - \frac{\sin x}{x} - \frac{\cos x}{x^2} \right) - B \left(\sin x + \frac{\cos x}{x} - \frac{\sin x}{x^2} \right),$$

$$\text{and } \frac{dA}{dx} \left(\sin x + \frac{\cos x}{x} \right) + \frac{dB}{dx} \left(\cos x - \frac{\sin x}{x} \right) = 0 \dots\dots(3).$$

$$\text{Also } \frac{dA}{dx} \left(\cos x - \frac{\sin x}{x} - \frac{\cos x}{x^2} \right) - \frac{dB}{dx} \left(\sin x + \frac{\cos x}{x} - \frac{\sin x}{x^2} \right) = x^2 \dots \dots (4).$$

From the equations (3) and (4), we proceed to find $\frac{dA}{dx}$, $\frac{dB}{dx}$,

$$\begin{aligned} \frac{dA}{dx} \left\{ \left(\cos x - \frac{\sin x}{x} \right)^2 - \frac{\cos x}{x^2} \left(\cos x - \frac{\sin x}{x} \right) + \left(\sin x + \frac{\cos x}{x} \right)^2 \right. \\ \left. - \frac{\sin x}{x^2} \left(\sin x + \frac{\cos x}{x} \right) \right\}, \\ = x^2 \left(\cos x - \frac{\sin x}{x} \right); \end{aligned}$$

$$\text{or } \frac{dA}{dx} \left(1 + \frac{1}{x^2} - \frac{1}{x^2} \right) = x^2 \left(\cos x - \frac{\sin x}{x} \right),$$

$$\therefore \frac{dA}{dx} = x^2 \left(\cos x - \frac{\sin x}{x} \right),$$

$$\text{and from (3) } \frac{dB}{dx} = -x^2 \left(\sin x + \frac{\cos x}{x} \right);$$

$$\therefore A = x^2 \sin x - 3 \int x \sin x dx,$$

$$= x^2 \sin x + 3x \cos x - 3 \sin x + C,$$

$$\text{and } B = x^2 \cos x - 3 \int x \cos x dx,$$

$$= x^2 \cos x - 3x \sin x - 3 \cos x + D;$$

therefore the complete integral of (2) is

$$\begin{aligned} \therefore y &= \left(\sin x + \frac{\cos x}{x} \right) (x^2 \sin x + 3x \cos x - 3 \sin x + C), \\ &+ \left(\cos x - \frac{\sin x}{x} \right) (x^2 \cos x - 3x \sin x - 3 \cos x + D), \\ &= x^2 + C \left(\sin x + \frac{\cos x}{x} \right) + D \left(\cos x - \frac{\sin x}{x} \right), \end{aligned}$$

C and D being arbitrary constants.

1849.

A curve is defined by this property, that the radius of curvature at any point in a given multiple of the portion of the

normal intercepted between the point and the axis of abscissæ; prove that the length of any portion of the curve may be expressed in finite terms of the ordinates of its extremities.

The lengths of the radius of curvature and normal are respectively

$$\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} \text{ and } y \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}};$$

hence the differential equation to the curve is

$$\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = ny \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}};$$

$$\therefore \frac{-\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{ny},$$

$$\text{or } \frac{\frac{d^2x}{dy^2}}{\frac{dx}{dy} \left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} = \frac{1}{ny}.$$

$$\text{Let } \frac{dx}{dy} = \tan \theta; \quad \therefore \frac{d^2x}{dy^2} = \left\{1 + \left(\frac{dx}{dy}\right)^2\right\} \frac{d\theta}{dy},$$

$$\text{and } \cot \theta \frac{d\theta}{dy} = \frac{1}{ny};$$

$$\therefore \log \cos \theta = -\frac{1}{n} \log Cy;$$

$$\text{or } \cos \theta = \frac{1}{(Cy)^{\frac{1}{n}}},$$

$$\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} \text{ or } \frac{dx}{dy} = (Cy)^{\frac{1}{n}},$$

$$\text{and } s = \frac{n}{n+1} C^{\frac{1}{n}} y^{\frac{n+1}{n}} + C',$$

C, C' being arbitrary constants.

Hence the length of any portion of the curve is known in terms of the ordinates of its extremities.

1850.

1. If $f(x-a, y-b, z-c)$ be homogeneous with respect to $x-a, y-b, z-c$, then $f(x-a, y-b, z-c) = 0$ is the equation of a cone whose vertex is (a, b, c) ; if the cone pass into a cylinder by a, b, c becoming infinite, shew algebraically that the limiting form of the above equation is

$$\phi(mx + ny + pz + q, m'x + n'y + p'z + q') = 0.$$

Let the axis of the cylinder, to which, as its limiting form, the cone tends as a, b, c , are indefinitely increased, be parallel to the intersection of the planes

$$mx + ny + pz = 0,$$

$$m'x + n'y + p'z = 0.$$

Then we have $ma + nb + pc =$ a finite quantity, α suppose,

$$m'a + n'b + p'c = \dots\dots\dots \alpha' \dots\dots\dots$$

Hence when a, b, c , become infinite we get, neglecting α, α' in comparison with a, b, c ,

$$\frac{a}{np' - n'p} = \frac{b}{pm' - p'm} = \frac{c}{mn' - m'n} \dots\dots\dots (1).$$

Now since f is a homogeneous function, we have, if n be its degree,

$$(x-a) \frac{df}{dx} + (y-b) \frac{df}{dy} + (z-c) \frac{df}{dz} = nf$$

$$= 0,$$

$$\therefore a \frac{df}{dx} + b \frac{df}{dy} + c \frac{df}{dz} = x \frac{df}{dx} + y \frac{df}{dy} + z \frac{df}{dz} \dots\dots (2).$$

Hence dividing each term of the left-hand member of (2) by the corresponding member of (1), and observing that when a, b, c , become infinite, the right-hand member will vanish after the division,

$$(np' - n'p) \frac{df}{dx} + (pm' - p'm) \frac{df}{dy} + (mn' - m'n) \frac{df}{dz} = 0 \dots (3) :$$

whence, by Lagrange's method, we get

$$\frac{dx}{np' - n'p} = \frac{dy}{pm' - p'm} = \frac{dz}{mn' - m'n};$$

$$\text{whence } mdx + ndy + pdz = 0,$$

$$m'dx + n'dy + p'dz = 0;$$

$$\therefore mx + ny + pz + q = 0,$$

$$m'x + n'y + p'z + q' = 0,$$

q, q' being constants.

Therefore the integral of (3) is

$$\phi(mx + ny + pz + q, m'x + n'y + p'z + q') = 0,$$

the limiting form of $f(x-a, y-b, z-c) = 0$, when a, b, c , are indefinitely increased.

2. Prove the following formulæ:

$$(1). \pi = 96 (2 + 2^{\frac{1}{2}}) \left(\frac{1}{1.3.5.7} + \frac{1}{9.11.13.15} + \dots \right).$$

$$(2). 1 + \frac{x^3}{1.2.3} + \frac{x^6}{1.2.3.4.5.6} + \dots = \frac{1}{3} \left(\epsilon^x + 2\epsilon^{-\frac{1}{2}x} \cos \frac{x.3^{\frac{1}{2}}}{2} \right).$$

$$(3). \frac{\cos\{(n-2r)a-s\}}{\sin(a-b)\sin(a-c)\dots} + \frac{\cos\{(n-2r)b-s\}}{\sin(b-a)\sin(b-c)\dots} + \dots \text{to } n \text{ terms} = 0,$$

where s is the sum of the n quantities a, b, c, \dots and r is any integer between 1 and $n-1$ inclusive.

(1). We have

$$\begin{aligned} & 48 \left(\frac{1}{1.3.5.7} + \frac{1}{9.11.13.15} + \dots \right) \\ &= 6 \left\{ \left(\frac{1}{1.7} - \frac{1}{3.5} \right) + \left(\frac{1}{9.15} - \frac{1}{11.13} \right) + \dots \right\} \\ &= \left\{ \left(\frac{1}{1} - \frac{1}{7} \right) - 3 \left(\frac{1}{3} - \frac{1}{5} \right) \right\} + \left\{ \left(\frac{1}{9} - \frac{1}{15} \right) - 3 \left(\frac{1}{11} - \frac{1}{13} \right) \right\} + \dots \\ &= 2 \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right) - \left(\frac{1}{1} + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \dots \right) \dots (1). \end{aligned}$$

$$\text{Now } \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{1}{4}\pi.$$

Again, generally

$$\frac{1}{1} + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots = \frac{1}{2x} \log \left(\frac{1+x}{1-x} \right).$$

Let $x^4 = -1$, then $x = \cos \frac{1}{4}\pi + -\frac{1}{2} \sin \frac{1}{4}\pi$,

$$\begin{aligned} \therefore \frac{1+x}{1-x} &= \frac{1 + \cos \frac{1}{4}\pi + -\frac{1}{2} \sin \frac{1}{4}\pi}{1 - \cos \frac{1}{4}\pi - -\frac{1}{2} \sin \frac{1}{4}\pi} \\ &= \frac{2 \cos^2 \frac{1}{8}\pi + -\frac{1}{2} 2 \sin \frac{1}{8}\pi \cos \frac{1}{8}\pi}{2 \sin^2 \frac{1}{8}\pi - -\frac{1}{2} 2 \sin \frac{1}{8}\pi \cos \frac{1}{8}\pi} \\ &= \cot \frac{1}{8}\pi \frac{\cos \frac{1}{8}\pi + -\frac{1}{2} \sin \frac{1}{8}\pi}{\sin \frac{1}{8}\pi - -\frac{1}{2} \cos \frac{1}{8}\pi} \\ &= -\frac{1}{2} \cot \frac{1}{8}\pi \\ &= \varepsilon^{-\frac{1}{2} \frac{1}{2}\pi} \cot \frac{1}{8}\pi; \end{aligned}$$

$$\therefore \log \left(\frac{1+x}{1-x} \right) = -\frac{1}{2} \frac{1}{2}\pi + \log \cot \frac{1}{8}\pi,$$

$$\begin{aligned} \text{and } \frac{1}{2x} &= \frac{1}{2^{\frac{1}{2}} + -\frac{1}{2} 2^{\frac{1}{2}}} \\ &= \frac{2^{\frac{1}{2}} - -\frac{1}{2} 2^{\frac{1}{2}}}{4}; \end{aligned}$$

$$\begin{aligned} \therefore \frac{1}{2x} \log \left(\frac{1+x}{1-x} \right) &= \frac{2^{\frac{1}{2}} - -\frac{1}{2} 2^{\frac{1}{2}}}{4} \left(-\frac{1}{2} \frac{1}{2}\pi + \log \cot \frac{1}{8}\pi \right) \\ &= \frac{1}{2 \cdot 2^{\frac{1}{2}}} \left(\log \cot \frac{1}{8}\pi + \frac{1}{2}\pi \right) - -\frac{1}{2} \frac{1}{2 \cdot 2^{\frac{1}{2}}} \left(\log \cot \frac{\pi}{8} - \frac{1}{2}\pi \right). \end{aligned}$$

Therefore, equating real and imaginary parts,

$$1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{1}{2 \cdot 2^{\frac{1}{2}}} \left(\log \cot \frac{1}{8}\pi + \frac{1}{2}\pi \right),$$

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \dots = -\frac{1}{2 \cdot 2^{\frac{1}{2}}} \left(\log \cot \frac{1}{8}\pi - \frac{1}{2}\pi \right);$$

$$\therefore \frac{1}{1} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots = \frac{\pi}{2 \cdot 2^{\frac{1}{2}}},$$

$$\begin{aligned} \therefore 48 \left(\frac{1}{1 \cdot 3 \cdot 5 \cdot 7} + \dots \right) &= \frac{\pi}{2} - \frac{\pi}{2 \cdot 2^{\frac{1}{2}}} \\ &= \frac{\pi}{2(2 + 2^{\frac{1}{2}})}, \end{aligned}$$

$$\pi = 96(2 + 2^{\frac{1}{2}}) \left(\frac{1}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{1}{9 \cdot 11 \cdot 13 \cdot 15} + \dots \right).$$

(2). In general,

$$\epsilon^y = 1 + \frac{y}{1} + \frac{y^2}{1.2} + \dots$$

Let $y = \omega x$, ω being one of the imaginary cube roots of unity,

$$\text{then } \epsilon^{\omega x} = 1 + \frac{\omega x}{1} + \frac{\omega^2 x^2}{1.2} + \dots$$

$$\text{Similarly, } \epsilon^{\omega^2 x} = 1 + \frac{\omega^2 x}{1} + \frac{\omega^3 x^2}{1.2} + \dots,$$

$$\epsilon^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \dots$$

$$\text{Now } 1 + \omega + \omega^2 = 0,$$

$$\therefore \epsilon^x + \epsilon^{\omega x} + \epsilon^{\omega^2 x} = 3 \left(1 + \frac{x^3}{1.2.3} + \dots \right).$$

$$\begin{aligned} \text{But } \epsilon^{\omega x} + \epsilon^{\omega^2 x} &= \epsilon^{(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)x} + \epsilon^{(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)x} \\ &= \epsilon^{-\frac{1}{2}x} \left(2 \cos \frac{\sqrt{3}x}{2} \right), \end{aligned}$$

$$\begin{aligned} \therefore 1 + \frac{x^3}{1.2.3} + \frac{x^6}{1.2.3.4.5.6} + \dots &= \frac{1}{3} (\epsilon^x + \epsilon^{\omega x} + \epsilon^{\omega^2 x}) \\ &= \frac{1}{3} \left(\epsilon^x + 2\epsilon^{-\frac{1}{2}x} \cos \frac{\sqrt{3}x}{2} \right). \end{aligned}$$

This problem may also be solved by putting the series $= u$, we shall then get the differential equation $\frac{d^3 u}{dx^3} - u = 0$, the integration of which, when the arbitrary constants are properly determined, will give the required value of u .

(3). If r lie between 1 and n , we may assume

$$\frac{\cos(n+1-2r)x}{\sin(x-a)\sin(x-b)\dots} = \frac{A}{\sin(x-a)} + \frac{B}{\sin(x-b)} + \dots,$$

A, B, \dots being quantities independent of x ,

$$\therefore \cos(n+1-2r)x = A \sin(x-b)\sin(x-c)\dots + B \sin(x-a)\sin(x-c)\dots + \dots \text{ identically.}^*$$

* We may justify the above assumption by expanding both sides of this equation in terms of $\sin x$ and $\cos x$, and dividing by $\cos^{n-1}x$: the left-hand side

Putting $x = a$, we get

$$\cos(n+1-2r)a = A \sin(a-b) \sin(a-c) \dots,$$

$$\therefore A = \frac{\cos(n+1-2r)a}{\sin(a-b) \sin(a-c) \dots} \dots\dots\dots (1).$$

$$\text{Similarly } B = \frac{\cos(n+1-2r)b}{\sin(b-a) \sin(b-c) \dots} \dots\dots\dots (2),$$

$$\dots = \dots\dots\dots;$$

$$\therefore \frac{\cos(n+1-2r)x}{\sin(x-a) \sin(x-b) \dots} = \frac{\cos(n+1-2r)a}{\sin(a-b) \sin(a-c) \dots \sin(x-a)}$$

$$+ \frac{\cos(n+1-2r)b}{\sin(b-a) \sin(b-c) \dots \sin(x-b)} + \dots;$$

$$\therefore \frac{\cos(n+1-2r)a}{\sin(a-b) \sin(a-c) \dots \sin(a-x)} + \frac{\cos(n+1-2r)b}{\sin(b-a) \sin(b-c) \dots \sin(b-x)} + \dots$$

$$+ \frac{\cos(n+1-2r)x}{\sin(x-a) \sin(x-b) \dots} = 0 \dots\dots (3).$$

In a similar manner it may be shewn that

$$\frac{\sin(n+1-2r)a}{\sin(a-b) \sin(a-c) \dots \sin(a-x)} + \frac{\sin(n+1-2r)b}{\sin(b-a) \sin(b-c) \dots \sin(b-x)} + \dots$$

$$+ \frac{\sin(n+1-2r)x}{\sin(x-a) \sin(x-b) \dots} = 0 \dots\dots (4).$$

(3) $\cos s$ + (4) $\sin s$, where $s = a + b + \dots + x$ gives

$$\frac{\cos\{(n+1-2r)a-s\}}{\sin(a-b) \sin(a-c) \dots} + \frac{\cos\{(n+1-2r)b-s\}}{\sin(b-a) \sin(b-c) \dots} + \dots = 0,$$

the required result proved for the $n+1$ quantities $a, b, \dots a_x$.

of the equation becomes $f(\tan x) \cdot (\sec x)^{2(r-1)}$, f being of $n-2r+1$ dimensions, or $f(\tan x) (1 + \tan^2 x)^{r-1}$, which is therefore of $n-1$ dimensions, and the equation becomes one of $n-1$ dimensions in $\tan x$; it may therefore be identically satisfied by the n quantities A, B, \dots . We here suppose $n+1-2r$ positive: if however $2r > n+1$, we may write for $2r, 2n+2-2s$, where $2s > 1 < n+1$; $\cos(n+1-2r)x$ then becomes $\cos(n+1-2s)x$, in which $n+1-2s$ is always positive.

3. Given $f(x) + f(y) = f(x+y) \{1 - f(x)f(y)\}$, find the form of $f(x)$.

Since $f(x) + f(y) = f(x+y) \{1 - f(x)f(y)\} \dots\dots\dots (1)$,
put $x = y = 0$, then

$$2f(0) = f(0) \{1 - \overline{f(0)}^2\};$$

therefore either $f(0) = 0$, or

$$1 - \overline{f(0)}^2 = 2,$$

$$\text{giving } f(0) = \pm (-1)^{\frac{1}{2}}.$$

Taking this latter value, and putting $y = 0$ in equation (1),

$$f(x) \pm (-1)^{\frac{1}{2}} = f(x) \{1 \mp (-1)^{\frac{1}{2}} f(x)\},$$

which gives $f(x) = \pm (-1)^{\frac{1}{2}}$ for all values of x ; therefore the given equation is satisfied by $f(x) = \pm (-1)^{\frac{1}{2}}$.

Again, if $f(0) = 0$, in equation (1) write $y = -x$, then

$$f(x) + f(-x) = f(0) \{1 - f(x)f(-x)\},$$

$$\therefore f(-x) = -f(x).$$

Differentiating (1) with respect to y , considering x constant,

$$f'(y) = f'(x+y) \{1 - f(x)f(y)\} - f(x+y) f(x) f'(y);$$

therefore, putting $y = -x$,

$$f'(-x) = f'(0) \{1 + \overline{f(x)}^2\};$$

or putting $-x = z$,

$$f'(z) = f'(0) \{1 + \overline{f(z)}^2\}.$$

Now $f'(0)$ = some constant, C suppose,

$$\therefore \frac{d.f(z)}{dz} = C \{1 + \overline{f(z)}^2\}.$$

Whence

$$f(z) = \tan Cz,$$

$$\text{or } f(x) = \tan Cx,$$

which determines the form of $f(x)$, C being an arbitrary constant.

1851.

1. Let P (fig. 98) be any point in a curve and S a given point in a straight line AS ; draw SU perpendicular to SP , and let the tangent at P meet SU , SA respectively in U and T ; find the nature of the curve when SU bears a constant ratio to ST .

Let the curve be referred to P as pole and SA as prime radius; then

$$SU : ST :: \sin STU : \sin SUT$$

$$:: \sin(\theta + SPU) : \cos SPU$$

$$:: \sin \theta + \cos \theta \tan SPU : 1;$$

$$\therefore \sin \theta + \cos \theta \tan SPU = \text{constant, } e \text{ suppose.}$$

$$\text{Now } \tan SPU = r \frac{d\theta}{dr},$$

$$\therefore r \cos \theta \frac{d\theta}{dr} = e - \sin \theta,$$

$$\text{or } \frac{\cos \theta}{e - \sin \theta} \frac{d\theta}{dr} = \frac{1}{r},$$

$$\text{and } \log(e - \sin \theta) = \log \frac{e}{r},$$

$$\text{or } \sin \theta = e - \frac{c}{r}.$$

Transforming this equation to rectangular coordinates,

$$y = e(x^2 + y^2)^{\frac{1}{2}} - c,$$

$$\text{or } e^2 x^2 + (e^2 - 1)y^2 - 2cy - c^2 = 0;$$

shewing that the curve is an ellipse or hyperbola according as e is $>$ or < 1 .

2. The equation $c - a \cos \theta \cos \phi - b \sin \theta \sin \phi = 0$ may be considered as the complete integral of

$$\frac{d\theta}{(a^2 \cos^2 \theta + b^2 \sin^2 \theta - c^2)^{\frac{1}{2}}} + \frac{d\phi}{(a^2 \cos^2 \phi + b^2 \sin^2 \phi - c^2)^{\frac{1}{2}}} = 0.$$

$$\text{If } c - a \cos \theta \cos \phi - b \sin \theta \sin \phi = 0 \dots\dots\dots(1),$$

$$(a \sin \theta \cos \phi - b \cos \theta \sin \phi) d\theta + (a \cos \theta \sin \phi - b \sin \theta \cos \phi) d\phi = 0,$$

and (1) may be considered as the complete integral of this equation, or of

$$\frac{d\theta}{a \cos \theta \sin \phi - b \sin \theta \cos \phi} + \frac{d\phi}{a \sin \theta \cos \phi - b \cos \theta \sin \phi} = 0 \dots(2),$$

c being considered an arbitrary constant.

$$\text{Now } (a \cos \theta \sin \phi - b \sin \theta \cos \phi)^2 + c^2 = (a \cos \theta \sin \phi - b \sin \theta \cos \phi)^2$$

$$+ (a \cos \theta \cos \phi + b \cos \theta \sin \phi)^2 \text{ by (1)}$$

$$= a^2 \cos^2 \theta + b^2 \sin^2 \phi,$$

$$\therefore a \cos \theta \sin \phi - b \sin \theta \cos \phi = (a^2 \cos^2 \theta + b^2 \sin^2 \phi - c^2)^{\frac{1}{2}};$$

and similarly,

$$a \sin \theta \cos \phi - b \cos \theta \sin \phi = (a^2 \cos^2 \phi + b^2 \sin^2 \theta - c^2)^{\frac{1}{2}},$$

therefore equation (2) becomes

$$\frac{d\theta}{(a^2 \cos^2 \phi + b^2 \sin^2 \theta - c^2)^{\frac{1}{2}}} + \frac{d\phi}{(a \sin \theta \cos \phi - b \cos \theta \sin \phi)^{\frac{1}{2}}} = 0.$$

DEFINITE INTEGRALS.

1849.

1. Shew that

$$\int_0^{\infty} \frac{x^{\frac{1}{2}} \log x}{(1+x)^2} dx = \pi.$$

Putting $x = \tan^2 \theta$, we get

$$\int_0^{\infty} \frac{x^{\frac{1}{2}} \log x}{(1+x)^2} dx = 4 \int_0^{\frac{1}{2}\pi} \sin^2 \theta \log \tan \theta d\theta.$$

Now generally $\int_0^x f(x) dx = \int_0^x f(a-x) dx$,

$$\begin{aligned} \therefore \int_0^{\frac{1}{2}\pi} \sin^2 \theta \log \tan \theta d\theta &= \int_0^{\frac{1}{2}\pi} \cos^2 \theta \log \cot \theta d\theta \\ &= - \int_0^{\frac{1}{2}\pi} \cos^2 \theta \log \tan \theta d\theta. \end{aligned}$$

Adding these equal quantities and dividing by 2,

$$\int_0^{\infty} \frac{x^{\frac{1}{2}} \log x}{(1+x)^2} dx = -2 \int_0^{\frac{1}{2}\pi} \cos 2\theta \log \tan \theta d\theta.$$

Now, integrating by parts,

$$- \int \cos 2\theta \log \tan \theta d\theta = -\frac{1}{2} \sin 2\theta \log \tan \theta + \theta + C,$$

$$\therefore - \int_0^{\frac{1}{2}\pi} \cos 2\theta \log \tan \theta d\theta = \frac{1}{2} \pi,$$

$$\therefore \int_0^{\infty} \frac{x^{\frac{1}{2}} \log x}{(1+x)^2} dx = \pi.$$

2. Shew that

$$\int_0^{\infty} \log x \log \left(\frac{x^2 + a^2}{x^2} \right) dx = \pi a (\log a - 1).$$

Let
$$\int_0^{\infty} \log x \log \left(\frac{x^2 + a^2}{x^2} \right) dx = u,$$

then
$$\frac{du}{da} = 2a \int_0^{\infty} \frac{\log x}{x^2 + a^2} dx;$$

put $x = a \tan \theta$, then

$$\frac{du}{da} = 2 \int_0^{\frac{1}{2}\pi} \log(a \tan \theta) d\theta \dots\dots\dots (1).$$

Now generally,
$$\int_0^x f(x) dx = \int_0^x f(a-x) dx,$$

$$\therefore \frac{du}{da} = 2 \int_0^{\frac{1}{2}\pi} \log(a \cot \theta) d\theta \dots\dots (2).$$

(1) + (2) gives, dividing by 2,

$$\frac{du}{da} = 2 \int_0^{\frac{1}{2}\pi} \log a d\theta$$

$$= \pi \log a,$$

$$\therefore u = \pi a (\log a - 1) + C.$$

And when $a = 0$, $u = 0$,

$$\therefore C = 0,$$

$$\text{and } u = \pi a (\log a - 1),$$

which was to be shewn.

3. Prove that

$$\int_0^{\pi} \frac{2e + (1 + e^2) \cos \theta}{(1 + 2e \cos \theta + e^2)^2} \log_{\epsilon} (1 + 2e \cos \theta + e^2) d\theta = \pm \pi \frac{e^{\pm 1}}{1 - e^2},$$

the upper or lower sign being taken according as e is less or greater than unity.

We have

$$\int \frac{2e + (1 + e^2) \cos \theta}{(1 + 2e \cos \theta + e^2)^2} d\theta = \frac{\sin \theta}{1 + 2e \cos \theta + e^2},$$

$$\therefore \int_0^{\pi} \frac{2e + (1 + e^2) \cos \theta}{(1 + 2e \cos \theta + e^2)^2} \log_{\epsilon} (1 + 2e \cos \theta + e^2) d\theta$$

$$= \frac{\sin \theta}{1 + 2e \cos \theta + e^2} \log(1 + 2e \cos \theta + e^2) + 2e \int_0^{\pi} \frac{\sin^2 \theta}{(1 + 2e \cos \theta + e^2)^2} d\theta$$

$$= \dots + \frac{\sin \theta}{1 + 2e \cos \theta + e^2} - \int_0^\pi \frac{\cos \theta}{1 + 2e \cos \theta + e^2}$$

$$= - \int_0^\pi \frac{\cos \theta d\theta}{1 + 2e \cos \theta + e^2}, \text{ between the limits,}$$

$$= - \frac{1}{2e} \int_0^\pi d\theta + \frac{1+e^2}{2e} \int_0^\pi \frac{d\theta}{1+e^2+2e \cos \theta}$$

$$= \frac{1+e^2}{2e} \int_0^\pi \frac{d\theta}{1+e^2+2e \cos \theta} - \frac{\pi}{2e}.$$

$$\text{And } \int_0^\pi \frac{d\theta}{1+e^2+2e \cos \theta}$$

$$= \int_0^\pi \frac{d\theta}{(1+e)^2 \cos^2 \frac{1}{2}\theta + (1-e)^2 \sin^2 \frac{1}{2}\theta} = 2 \int_0^\infty \frac{d \tan \frac{1}{2}\theta}{(1+e)^2 + (1-e)^2 \tan^2 \frac{1}{2}\theta}$$

$$= \frac{2}{1-e^2} \tan^{-1} \frac{1-e}{1+e} \tan \frac{1}{2}\pi = \frac{\pi}{1-e^2} \text{ if } e < 1$$

$$= \frac{2}{e^2-1} \tan^{-1} \frac{e-1}{e+1} \tan \frac{1}{2}\pi = \frac{\pi}{e^2-1} \text{ if } e > 1;$$

therefore the definite integral becomes in the two cases

$$\begin{aligned} \pm \frac{1+e^2}{2e} \frac{\pi}{1-e^2} - \frac{\pi}{2e} &= \pi \frac{2e^2}{2e(1-e^2)} \text{ and } -\pi \frac{2}{2e(1-e^2)} \\ &= \pm \pi \frac{e^{\pm 1}}{1-e}. \end{aligned}$$

1850.

1. Shew that

$$(\alpha). \quad \int_0^{\frac{1}{2}\pi} \tan^{\frac{1}{3}} x \log(\tan x) dx = \frac{1}{6} \pi^2,$$

and that

$$(\beta). \quad \int_{-a}^{+a} \frac{(a^2 - x^2)^{\frac{1}{2}}}{\frac{a}{\sin 2\varepsilon} - x} dx = \pi a \tan \varepsilon \text{ or } \pi a \cot \varepsilon,$$

according as ε is $<$ or $> \frac{1}{4}\pi$.

(α). We have, a being < 1 ,

$$\int_0^\infty \frac{z^{a-1}}{1+z} dz = \frac{\pi}{\sin a\pi}.$$

(See Gregory's *Examples*, p. 477.)

Differentiating with respect to a ,

$$\int_0^{\infty} \frac{z^{a-1}}{1+z} \log z \, dz = -\frac{\pi^2 \cos a\pi}{\sin^2 a\pi}.$$

Putting $z = \tan^2 x$, the limits of x will be $0, \frac{1}{2}\pi$, hence

$$4 \int_0^{\frac{1}{2}\pi} \tan^{2a-1} x \log(\tan x) \, dx = -\frac{\pi^2 \cos a\pi}{\sin^2 a\pi},$$

whence, putting $a = \frac{2}{3}$,

$$4 \int_0^{\frac{1}{2}\pi} \tan^{\frac{1}{3}} x \log(\tan x) \, dx = \frac{\pi^2 \cos \frac{2\pi}{3}}{\sin^2 \frac{2\pi}{3}} = \frac{2\pi^2}{3};$$

$$\therefore \int_0^{\frac{1}{2}\pi} \tan^{\frac{1}{3}} x \log(\tan x) \, dx = \frac{1}{6} \pi^2.$$

(β). Putting $x = a \sin \theta$, we have

$$\int_{-a}^{+a} \frac{(a^2 - x^2)^{\frac{1}{2}}}{\frac{a}{\sin 2\varepsilon} - x} \, dx = a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2 \theta}{\frac{1}{\sin 2\varepsilon} - \sin \theta} \, d\theta;$$

therefore writing $-\theta$ for θ ,

$$= a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2 \theta}{\frac{1}{\sin 2\varepsilon} + \sin \theta} \, d\theta.$$

Adding these equals and dividing by 2,

$$\begin{aligned} \int_{-a}^{+a} \frac{(a^2 - x^2)^{\frac{1}{2}}}{\frac{a}{\sin 2\varepsilon} - x} \, dx &= \frac{a}{\sin 2\varepsilon} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2 \theta}{\frac{1}{\sin^2 2\varepsilon} - \sin^2 \theta} \, d\theta, \\ &= \frac{a}{\sin 2\varepsilon} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left(1 - \frac{\cos^2 2\varepsilon}{(1 - \sin^2 2\varepsilon \sin^2 \theta)} \right) d\theta, \\ &= \frac{\pi a}{\sin 2\varepsilon} - a \frac{\cos^2 2\varepsilon}{\sin 2\varepsilon} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2 \theta}{1 + \cos^2 2\varepsilon \tan^2 \theta} \, d\theta. \end{aligned}$$

$$\text{Now } \int \frac{\sec^2 \theta}{1 + \cos^2 2\varepsilon \tan^2 \theta} \, d\theta = \sec 2\varepsilon \tan^{-1}(\cos 2\varepsilon \tan \theta) + C;$$

$$\therefore \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2 \theta}{1 + \cos^2 2\varepsilon \tan^2 \theta} \, d\theta = \pi \sec 2\varepsilon;$$

$$\begin{aligned}\therefore \int_{-a}^{+a} \frac{(a^2 - x^2)^{\frac{1}{2}}}{\frac{a}{\sin 2\varepsilon} - x} dx &= \pi a \left(\frac{1}{\sin 2\varepsilon} - \frac{\cos 2\varepsilon}{\sin 2\varepsilon} \right) \\ &= \pi a \tan \varepsilon.\end{aligned}$$

The above investigation holds if $\varepsilon < \frac{1}{4}\pi$. If $\varepsilon > \frac{1}{4}\pi$, let $\varepsilon = \frac{1}{2}\pi - \varepsilon'$ (ε' being $< \frac{1}{4}\pi$), then

$$\sin 2\varepsilon = \sin 2\varepsilon',$$

$$\begin{aligned}\text{and } \int_{-a}^{+a} \frac{(a^2 - x^2)^{\frac{1}{2}}}{\frac{a}{\sin 2\varepsilon'} - x} dx &= \pi a \tan \varepsilon', \\ &= \pi a \cot \varepsilon;\end{aligned}$$

$$\therefore \int_{-a}^{+a} \frac{(a^2 - x^2)^{\frac{1}{2}}}{\frac{a}{\sin 2\varepsilon} - x} dx = \pi a \tan \varepsilon \text{ or } \pi a \cot \varepsilon,$$

according as $\varepsilon < \text{or} > \frac{1}{4}\pi$.

1851.

1. If y be a function of x defined by the equation

$$\alpha^{2n} = (y - nx)^{n+\beta} (y + nx)^{n-\beta},$$

$$\text{shew that } \int_0^x \frac{dx}{y + \beta x} = \int_a^y \frac{dy}{\beta y + n^2 x} = \frac{1}{n + \beta} \log \frac{y + nx}{\alpha}.$$

$$\text{Since } \alpha^{2n} = (y - nx)^{n+\beta} (y + nx)^{n-\beta};$$

therefore, taking the logarithmic differential,

$$\begin{aligned}0 &= (n + \beta) \frac{dy - n dx}{y - nx} + (n - \beta) \frac{dy + n dx}{y + nx} \\ &= \frac{1}{y^2 - n^2 x^2} [(n + \beta)(y + nx) + (n - \beta)(y - nx)] dy \\ &\quad + n \{(n - \beta)(y - nx) - (n + \beta)(y + nx)\} dx \\ &= \frac{2n}{y^2 - n^2 x^2} \{(y + \beta x) dy - (\beta y + n^2 x) dx\}; \\ \therefore \frac{dx}{y + \beta x} &= \frac{dy}{\beta y + n^2 x}; \\ \therefore \int \frac{dx}{y + \beta x} &= \int \frac{dy}{\beta y + n^2 x},\end{aligned}$$

between proper limits of the variables. Now when $x = 0$, $y = \alpha$,

$$\begin{aligned}\therefore \int_0^x \frac{dx}{y + \beta x} &= \int_\alpha^y \frac{dy}{\beta y + n^2 x}, \\ &= \int_\alpha^{y+nx} \frac{d(y+nx)}{\beta y + n^2 x + ny + n\beta x}, \\ &= \int_\alpha^{y+2nx} \frac{d(y+nx)}{(n+\beta)(y+nx)}, \\ &= \frac{1}{n+\beta} \log \frac{y+nx}{\alpha},\end{aligned}$$

the required result.

2. Determine the value of the definite integral

$$\int_0^1 \frac{x^{x-1} (1-x)^{\beta-1} dx}{(x+a)^{\alpha+\beta}}.$$

Let

$$\frac{x}{x+a} = \frac{y}{1+a},$$

then when $x = 0$, $y = 0$, and when $x = 1$, $y = 1$,

$$x = \frac{ay}{1+a-y};$$

$$\therefore 1-x = \frac{(1+a)(1-y)}{1+a-y},$$

$$x+a = a \frac{1+a}{1+a-y},$$

$$\left(\frac{1-x}{x+a}\right)^\beta = \frac{1}{a^\beta} (1-y)^\beta;$$

$$\therefore \frac{(1-x)^{\beta-1}}{(x+a)^\beta} = \frac{(1-y)^{\beta-1}}{a^\beta} \times \frac{1+a-y}{1+a}.$$

Also $\frac{x^x}{(x+a)^\alpha} = \frac{y^x}{(1+a)^\alpha},$

$$\frac{dx}{x} = \frac{dy}{y} + \frac{dy}{1+a-y},$$

$$= \frac{1+a}{y(1+a-y)} dy;$$

$$\therefore \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(x+a)^{\alpha+\beta}} dx = \frac{1}{a^{\beta}(1+a)^{\alpha}} y^{\alpha-1}(1-y)^{\beta-1} dy ;$$

$$\begin{aligned} \therefore \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(x+a)^{\alpha+\beta}} dx &= \frac{1}{a^{\beta}(1+a)^{\alpha}} \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy, \\ &= \frac{1}{a^{\beta}(1+a)^{\alpha}} \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}, \end{aligned}$$

the required value.*

* See *Gregory's Examples*, p. 471.

CALCULUS OF FINITE DIFFERENCES.

1851.

1. PT, pt , (fig. 90) are two tangents to a curve drawn at the extremities of any chord PSp passing through the pole S , and meeting a given line SA in T, t , respectively; it is required to prove that the curve in which the sum of the reciprocals of ST and St is constant has for its equation

$$\frac{a}{r} = 1 + e \cos \theta + f(\sin \theta)^2,$$

where $f(\sin \theta)^2$ denotes any rational function of $(\sin \theta)^2$.

Let the angle $SPT = \phi$, $PST = \theta$, $SP = r$, then

$$\begin{aligned} \frac{r}{ST} &= \frac{\sin(\theta + \phi)}{\sin \phi}, \\ &= \sin \theta \cot \phi + \cos \theta, \\ &= \sin \theta \frac{1}{r} \frac{dr}{d\theta} + \cos \theta; \end{aligned}$$

$$\therefore \frac{1}{ST} = -\sin \theta \frac{du_{\theta}}{d\theta} + u_{\theta} \cos \theta,$$

putting $u_{\theta} = \frac{1}{r}$.

Similarly, writing $\theta + \pi$ for θ ,

$$\begin{aligned} \frac{1}{St} &= -\sin(\theta + \pi) \frac{du_{\theta+\pi}}{d\theta} + u_{\theta+\pi} \cos(\theta + \pi), \\ &= \sin \theta \frac{du_{\theta+\pi}}{d\theta} - u_{\theta+\pi} \cos \theta; \end{aligned}$$

$$\begin{aligned} \therefore \frac{1}{ST} + \frac{1}{St} &= \sin \theta \left(\frac{du_{\theta+\pi}}{d\theta} - \frac{du_{\theta}}{d\theta} \right) - \cos \theta (u_{\theta+\pi} - u_{\theta}), \\ &= \sin^2 \theta \frac{d}{d\theta} \left(\frac{u_{\theta+\pi} - u_{\theta}}{\sin \theta} \right). \end{aligned}$$

Now by the conditions of the problem,

$$\frac{1}{ST} + \frac{1}{St} = a \text{ constant, } c \text{ suppose ;}$$

$$\therefore \frac{d}{d\theta} \left(\frac{u_{\theta+\pi} - u_{\theta}}{\sin \theta} \right) = \frac{c}{\sin^2 \theta},$$

$$\therefore u_{\theta+\pi} - u_{\theta} = \sin \theta (b - c \cot \theta),$$

b being an arbitrary constant,

$$= b \sin \theta - c \cos \theta,$$

$$= -\frac{2c}{a} \cos(\theta + \alpha), \text{ changing the constants.}$$

But by measuring θ from a proper point, we shall get $\alpha = 0$, so that we may write

$$u_{\theta+\pi} - u_{\theta} = -\frac{2c}{a} \cos \theta,$$

an equation of differences, which, when integrated, gives

$$u_{\theta} = \frac{e}{a} \cos \theta + C_{\theta},$$

C_{θ} being any function of θ which does not change its value when $\pi + \theta$ is written for θ .

Therefore we may put

$$C_{\theta} = \frac{1 + f(\sin \theta)^2}{a},$$

and our equation becomes

$$u_{\theta} = \frac{1 + e \cos \theta + f(\sin \theta)^2}{a},$$

or, since $u_{\theta} = \frac{1}{r}$,

$$\frac{a}{r} = 1 + e \cos \theta + f(\sin \theta)^2.$$

The following Problem in Geometry of Three Dimensions (set in 1851) has been omitted.

Determine the surface generated by a tangent to a right cylinder which moves parallel to the base, and with its point

of contact lying on a helix: shew also that a hyperboloid of one sheet may be constructed touching the cylinder along its base, and such that the required surface and the hyperboloid are developable, the one on the other.

(α) Take the axis of the cylinder as that of z : let a be its radius, and let the equations to the helix be

$$x = a \cos \frac{z}{c}, \quad y = a \sin \frac{z}{c}.$$

Let ξ, η, ζ , be the current coordinates of the required surface, the equations to the generating line touching the cylinder at (x, y, z) will be

$$\xi \cos \frac{z}{c} + \eta \sin \frac{z}{c} = a, \quad \zeta = z.$$

Eliminating z between these equations, we get

$$\xi \cos \frac{\zeta}{c} + \eta \sin \frac{\zeta}{c} = a$$

as the equation to the required surface.

(β) It is obvious that the inclinations of a generating line of the cylinder to a tangent to the helix and to a generating line of the hyperboloid, are respectively constant, and that the arbitrary parameter of the hyperboloid may be so determined as to make these two angles equal to one another, each equal the angle ι suppose. Let α, α' be two points on the base of the cylinder, indefinitely near together, $\alpha\beta, \alpha'\beta'$ the generating lines of the hyperboloid passing through them. Also let A, A' be two points on the helix, such that the elementary arc AA' equals the elementary arc $\alpha\alpha'$; and let $AB, A'B'$ be the generating lines of the helicoidal surface passing through A, A' respectively, and make $\alpha\beta = \alpha'\beta' = AB = A'B'$.

We may then shew that $BB' = \beta\beta'$, *whatever be the magnitude of $\alpha\beta$* . Therefore, if the circle in which the hyperboloid touches the cylinder be laid upon the helix in which the helicoidal surface touches it, the element of surface between two

consecutive generating lines of the former figure may be superimposed on the corresponding element in the latter figure, and so for the other elements; the flexure only taking place round the consecutive generating lines. Hence the two surfaces are developable, the one on the other.

We proceed to shew that $BB' = \beta\beta'$, as above stated.

It is easy to see that $2\pi c$ is the distance between the successive threads of the helix, and therefore $\frac{c}{a} = \cot \iota$: hence the equation to the required hyperboloid is

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1.$$

It is manifest that β, β' lie upon the same circular section of the hyperboloid; the radius (r) of this section, whose altitude call z ,

$$\begin{aligned} r &= a \left(1 + \frac{z^2}{c^2} \right)^{\frac{1}{2}} \text{ by the above equation} \\ &= a \left(1 + \frac{\alpha\beta^2 \sin^2 \iota}{c^2} \right)^{\frac{1}{2}}. \end{aligned}$$

Also $\beta\beta'$ subtends at the centre of its section the same angle that $\alpha\alpha'$ does at the centre of the base of the cylinder;

$$\begin{aligned} \therefore \beta\beta' &= \frac{r}{a} \alpha\alpha' \\ &= \left(1 + \frac{\alpha\beta^2 \sin^2 \iota}{c^2} \right)^{\frac{1}{2}} \alpha\alpha'. \end{aligned}$$

Again, B and B' lie on the surface of a cylinder with the same axis as the proposed, and whose radius (r')

$$= (a^2 + AB^2)^{\frac{1}{2}}.$$

Also BB' is the same part of the thread of a helix on this cylinder that AA' is of the given helix; the helices having the same distances $2\pi c$ between the threads, and therefore their lengths being

$$\begin{aligned} &\{(2\pi r')^2 + (2\pi c)^2\}^{\frac{1}{2}} \text{ and } \{(2\pi a)^2 + (2\pi c)^2\}^{\frac{1}{2}}, \\ \text{or } &2\pi (a^2 + AB^2 + c^2)^{\frac{1}{2}} \text{ and } 2\pi (a^2 + c^2)^{\frac{1}{2}}; \end{aligned}$$

$$\begin{aligned}
 \therefore BB' &= \left(1 + \frac{AB^2}{a^2 + c^2}\right)^{\frac{1}{2}} AA' \\
 &= \left(1 + \frac{AB^2 \sin^2 \iota}{a^2}\right)^{\frac{1}{2}} AA', \quad \because c = a \cot \iota, \\
 &= \left(1 + \frac{\alpha \beta^2 \sin^2 \iota}{a^2}\right)^{\frac{1}{2}} AA';
 \end{aligned}$$

therefore $BB' = bb'$, and the surfaces are developable, the one on the other.*

* We are indebted to Mr. Cayley for the solution of the second part of this problem.

STATICS.

1848.

1. A uniform slender rod passes over the fixed point A (fig. 91) and under the fixed point B , and is kept at rest by the friction at the points A and B : determine the limiting positions of equilibrium.

It is evident that the friction must always act upwards, and the limiting position of equilibrium will be one in which G is so near A , that the friction is only just able to support the resolved part of the weight along the rod.

Let $AB = a$, $AG = x$, when G is in the limiting position.

Resolving the forces on the rod along it and perpendicular to its length,

$$\mu R + \mu R' = W \cos \alpha,$$

$$R - R' = W \sin \alpha;$$

and taking moments about G ,

$$R'(a + x) = Rx;$$

whence the above equations become

$$\mu R \left(1 + \frac{x}{a + x} \right) = W \cos \alpha,$$

$$R \left(1 - \frac{x}{a + x} \right) = W \sin \alpha.$$

Whence, by division,

$$\mu \frac{a + 2x}{a} = \cot \alpha,$$

$$\text{and } x = \frac{1}{2} \left(\frac{\cot \alpha}{\mu} - 1 \right) a,$$

which is the least possible value for x : G may be as high above it as is consistent with the leaning of the rod against B .

If A were lower than B , G might be as low as we please, but at no less distance from B than the above value of x .

2. Four uniform slender rods, AB , BC , CD , DA , (fig. 92), rigidly connected, form the sides of a quadrilateral figure, such that the angle A is a right angle, and the points B , C , D , are equidistant from each other: when the whole is suspended at the angle A , determine the position of equilibrium.

Let \bar{x} , \bar{y} be the coordinates of the centre of gravity of the system referred to AB , AD as coordinate axes; and let $AB = 2a$, $AD = 2b$, and the angle $ABD = \alpha$;

$$\begin{aligned} \therefore \{2a + 2b + 4(a^2 + b^2)^{\frac{1}{2}}\} \bar{x} \\ &= 2a \cdot a + 2(a^2 + b^2)^{\frac{1}{2}} [2a + (a^2 + b^2)^{\frac{1}{2}} \{\cos(120^\circ - \alpha) - \cos(120^\circ + \alpha)\}] \\ &= 2a^2 + 2(a^2 + b^2)^{\frac{1}{2}} \{2a + (a^2 + b^2)^{\frac{1}{2}} 3^{\frac{1}{2}} \sin \alpha\} \\ &= 2a^2 + 2(a^2 + b^2)^{\frac{1}{2}} (2a + 3^{\frac{1}{2}} \cdot b). \end{aligned}$$

Similarly,

$$\{2a + 2b + 4(a^2 + b^2)^{\frac{1}{2}}\} \bar{y} = 2b^2 + 2(a^2 + b^2)^{\frac{1}{2}} (2b + 3^{\frac{1}{2}} a).$$

Hence, if θ be the inclination of AB to the vertical,

$$\tan \theta = \frac{\bar{y}}{\bar{x}} = \frac{b^2 + (a^2 + b^2)^{\frac{1}{2}} (2b + 3^{\frac{1}{2}} a)}{a^2 + (a^2 + b^2)^{\frac{1}{2}} (2a + 3^{\frac{1}{2}} b)}.$$

3. A string of given length is attached to the extremities of the arms of a straight lever without weight, and passes round a small pulley which supports a weight: find the position of equilibrium in which the lever is inclined to the vertical, and prove that the equilibrium is unstable.

The inclination of the lever to the horizon will be determined in this case in the method to be shewn in the next problem but one, the point G being now the fulcrum, and P vertically below G instead of above it.

To determine whether the equilibrium is stable or unstable, let the lever be turned through a small angle; then the weight will assume the lowest position it can, and the normal at this point to the ellipse mentioned in the above problem will be vertical.

Hence it is evident that the vertical, through the weight in its displaced position, will intersect the lever on that side of the fulcrum which is lowered in the above arbitrary displacement: hence the system will tend further from its position of rest, and the equilibrium is unstable.

4. Two equal strings, of length l , are attached to the fixed points A, B , and C, D , respectively, which, if joined, would form a horizontal rectangle; a sphere, whose diameter equals AB , is laid symmetrically upon the strings: find the position of equilibrium and the tension of either string, supposing

$$l > AC + \frac{1}{2}\pi AB.$$

Shew also how the problem is to be solved when this condition is not fulfilled.

The centre of the sphere must lie in the vertical line through the point of intersection of the diagonals of the parallelogram $ABCD$, and each string must lie wholly in the plane through its points of support and the centre of the sphere.

Let $AB = 2a$, $AC = 2b$, and the depth of the centre of the sphere $= z$;

$$\therefore l = 2(b^2 + z^2)^{\frac{1}{2}} + \pi a,$$

$$\text{or } z = \left\{ \frac{1}{4}(l - \pi a)^2 - b^2 \right\}^{\frac{1}{2}} \dots\dots\dots (1).$$

Let T equal the tension of either string;

$$\therefore 4T \frac{z}{(b^2 + z^2)^{\frac{1}{2}}} = \text{weight of the sphere} \dots\dots (2):$$

from equations (1) and (2) T is known.

If $l = 2b + \pi a$, $z = 0$ and $T = \infty$, the centre of the sphere being in the horizontal plane $ABCD$.

If $l < 2b + \pi a$, we must suppose the part of the string not in contact with the sphere to become rigid, so as to support the sphere above the horizontal plane $ABCD$. Each string must still lie wholly in the plane through its points of support and the centre of the sphere.

1849.

1. Two unequal weights, connected by a straight rod without weight, are suspended by a string fastened at the extremities of the rod, and passing over a fixed point: determine the position of equilibrium.

Let G (fig. 93) be the centre of gravity of W and W' , the two weights, P the pulley: PG must be vertical, and bisect the angle WPW' .

Let $WPIW' = 2l$, $WW' = 2a$: then PG is the normal of the ellipse, which has WW' for foci and $2l$ for axis-major; hence

$$WP : WG :: WPIW' : WW',$$

$$\text{or } WP = \frac{W'}{W + W'} 2l,$$

$$\text{and } W'P = \frac{W}{W + W'} 2l;$$

hence the sides of the triangle WPW' are known, and thence its angles: therefore $\angle WPG = \frac{1}{2} WPIW'$ is known, and $WGP = \pi - WPG - PWG$ is known, which is the inclination of WW' to the vertical.

2. A smooth body, in the form of a sphere, is divided into hemispheres, and placed with the plane of division vertical upon a smooth horizontal plane: a string, loaded at its extremities with two equal weights, hangs upon the sphere, passing over its highest point, and cutting the plane of division at right angles: find the least weight which will preserve the equilibrium. Determine whether the equilibrium is stable or unstable.

Let a = radius of the sphere;

\bar{x} = distance of the centre of gravity of the hemisphere from the plane of division;

W = weight of the sphere;

w = weight required.

We may consider the string to become rigidly attached to the sphere without disturbing the equilibrium: we then have

each system of a hemisphere and weight attached prevented from turning about the line of intersection of the horizontal plane and plane of division, by the tension w , at the highest point of the hemisphere.

Hence, taking moments about this line,

$$w.2a = W\bar{x} + wa;$$

$$\therefore w = \frac{\bar{x}}{a} W = \frac{3}{8} W \dots\dots\dots (1).$$

To consider whether the equilibrium is stable or unstable.

If we give either hemisphere a small angular displacement (θ) about the above line, the weight w rises through a space $a\theta$, and the centre of gravity of W falls through a space $\bar{x}.\theta$. Hence the common centre of gravity of the hemisphere and weight rises through a space

$$wa\theta - W\bar{x}\theta = 0, \text{ by (1),}$$

and the equilibrium is therefore neuter.

3. A slightly elastic string, attached to two points in the circumference of the base of a right cone, at opposite extremities of a diameter, is just long enough to reach over the vertex without stretching. The cone is suspended by it from its middle point: find approximately the increase of its length.

Let $2l$ = unstretched length of the string;

h = height of the cone;

a = radius of its base;

z = the depth through which the cone falls;

$2(l + \lambda)$ = the stretched length of the string.

Then, by the principle that "tension varies as extension", if T be the tension of the string,

$$T = E \frac{\lambda}{l}, \quad E \text{ a constant weight};$$

and $2T \frac{h+z}{l+\lambda} = W$ the weight of the cone.

$$\text{Also, } (h+z)^2 = (l+\lambda)^2 + a^2;$$

$$\therefore 2E \frac{\lambda}{l} \frac{\{(l+\lambda)^2 + a^2\}^{\frac{1}{2}}}{l+\lambda} = W:$$

or, omitting λ^2 and the higher powers of λ ,

$$2E \frac{\lambda}{l} \frac{(l^2 + a^2)^{\frac{1}{2}}}{l} = W,$$

$$\text{and } \lambda = \frac{W}{2E} \frac{l^2}{(l^2 - a^2)^{\frac{1}{2}}}.$$

4. An equilateral triangle, without weight, has three unequal particles placed at its angular points; the system is suspended from a fixed point by three equal strings at right angles to each other, and fastened to the corners of the triangle: find the inclination of the plane of the triangle to the horizon.

Let \bar{x} , \bar{y} , \bar{z} , be the coordinates of the centre of gravity of the three weights referred to the strings as axes: \bar{x} , \bar{y} , \bar{z} , will be subject to the condition

$$\bar{x} + \bar{y} + \bar{z} = l,$$

if l be the length of the strings.

Let θ be the angle between the normal to the plane of the triangle and the line joining the centre of gravity with the origin, which is vertical; this angle will be the required inclination of the plane to the horizon. The direction-cosines

of these lines are $\frac{1}{3^{\frac{1}{2}}}$, $\frac{1}{3^{\frac{1}{2}}}$, $\frac{1}{3^{\frac{1}{2}}}$, and $\frac{\bar{x}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{\frac{1}{2}}}$, $\frac{\bar{y}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{\frac{1}{2}}}$, $\frac{\bar{z}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{\frac{1}{2}}}$, respectively;

$$\begin{aligned} \therefore \cos \theta &= \frac{1}{3^{\frac{1}{2}}} \cdot \frac{\bar{x} + \bar{y} + \bar{z}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{\frac{1}{2}}} \\ &= \frac{1}{3^{\frac{1}{2}}} \cdot \frac{l}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{\frac{1}{2}}}. \end{aligned}$$

5. A piece of string is fastened at its extremities to two fixed points: determine from mechanical considerations the form which must be assumed by the string in order that the surface generated by its revolution about the line joining the fixed points may be the greatest possible.

By Guldinus' property of the centre of gravity, that curve will by its revolution generate the greatest surface whose centre of gravity is furthest from the axis, *i.e.* is lowest, when the axis is made horizontal and the plane of the curve vertical. Now we know the centre of gravity will assume the lowest possible position when the string is in equilibrium under the action of gravity: hence the curve required is the common catenary.

6. It is required to support a smooth heavy body, in the form of an ellipsoid, in such a manner, that a given radius in the body shall be vertical, by means of supports at three points: shew that if l , m , n , be the direction-cosines of the radius, and the equation of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

then the three points in question must be on the curve of intersection of the ellipsoid with the cone

$$lyz\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + mzx\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + mxy\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0.$$

We will assume the normals at the three points to meet in some point of the vertical radius.*

The equation to the normal at x_1, y_1, z_1 , is

$$\frac{\xi - x_1}{\frac{x_1}{a^2}} = \frac{\eta - y_1}{\frac{y_1}{b^2}} = \frac{\zeta - z_1}{\frac{z_1}{c^2}},$$

* This is an assumption: for the ellipsoid will be supported if two of the normals meet in a point not in the vertical radius, provided the resultant of the corresponding reactions meet the vertical radius in the same point as the third normal does.

and this line passes through the points lr , mr , nr ;

$$\therefore \frac{lr - x_1}{\frac{x_1}{a^2}} = \frac{mr - y_1}{\frac{y_1}{b^2}} = \frac{nr - z_1}{\frac{z_1}{c^2}} = s \text{ suppose:}$$

with similar equations for the coordinates of the other points of support.

These equations may be written

$$lr - \frac{x_1}{a^2} s = x_1,$$

$$mr - \frac{y_1}{b^2} s = y_1,$$

$$nr - \frac{z_1}{c^2} s = z_1;$$

whence, eliminating r and s by cross-multiplication,

$$x_1 \left(\frac{mz_1}{c^2} - \frac{ny_1}{b^2} \right) + y_1 \left(\frac{nx_1}{a^2} - \frac{lz_1}{c^2} \right) + z_1 \left(\frac{ly_1}{b^2} - \frac{mx_1}{a^2} \right) = 0;$$

or, dropping the suffix, we have

$$lyz \left(\frac{1}{b^2} - \frac{1}{c^2} \right) + mzx \left(\frac{1}{c^2} - \frac{1}{a^2} \right) + nxy \left(\frac{1}{a^2} - \frac{1}{b^2} \right) = 0,$$

as the equation to the cone on which the three points of support must lie.

1850.

1. A right cone is cut obliquely, and then placed with its section on a horizontal plane: prove that, when the angle of the cone is less than $\sin^{-1} \frac{1}{4}$, there will be two sections for which the equilibrium is neutral, and for intermediate sections the cone will fall over.

Let ABC (fig. 94) be the section of the cone through its axis, by the plane of the paper, to which the cutting plane is supposed perpendicular. Let the trace BP of the cutting plane make an angle θ with BC : draw PD perpendicular to BP , and draw AEF through F , the bisection of BP .

Let 2α be the angle of the cone; then $\angle ABP = \pi - \alpha - \theta$, $\angle BDP = \theta + \alpha$, and $\angle APD = \theta - \alpha$.

Also, let $AE = n.EF$, then

$$\begin{aligned} n = \frac{AE}{EF} &= \frac{AP \sin APE}{PF \sin BPD} = \frac{2AP \sin(\theta - \alpha)}{BP} \\ &= \frac{2 \cos(\theta + \alpha) \sin(\theta - \alpha)}{\sin 2\alpha}, \end{aligned}$$

$$\text{or } n \sin 2\alpha = \sin 2\theta - \sin 2\alpha;$$

$$\therefore \sin 2\theta = (n + 1) \sin 2\alpha.$$

If $n = 3$, E will be the centre of gravity of the part cut off, which will therefore stand on its base in neutral equilibrium, and

$$\sin 2\theta = 4 \sin 2\alpha.$$

Hence, if $\sin 2\alpha < \frac{1}{4}$, there will be two values of θ , each acute, such that the corresponding cutting planes shall give neutral equilibrium. For intermediate sections,

$$\sin 2\theta > 4 \sin 2\alpha,$$

and therefore $n > 4$;

hence the centre of gravity will lie outside the vertical line PD , and the section will fall over.

2. The three corners of a triangle are kept on a circle by three rings capable of sliding along the circle, and the circle is inclined to the horizon at a given angle: find the positions of equilibrium.

It is evident that, as the triangle is moved about, its centre of gravity describes a circle about the centre of the circle, the positions of equilibrium are those in which the centre of gravity is at the lowest and highest points respectively of this circle. The corresponding positions of the triangle are easily found.

3. A smooth cylinder is supported in a position of equilibrium by a string which is wound m times round it, and then has its extremities attached to two points A and B in the same horizontal line. The position of equilibrium being that in which the coils are separate, shew how it is determined,

and how to find the length of the string in contact with the cylinder.

Produce the straight lines of the string to meet, as they must do, in a point: project these produced parts and the curved part of the string upon the axis of the cylinder; these projections must be equal; but the inclination to the axis of the straight and curved parts of the string is the same; hence the produced parts of the string must be equal in length to the part in contact with the cylinder. Hence the straight parts of the string occupy the same position as they would do if the string, instead of supporting the cylinder, ran under an indefinitely small pulley supporting a weight. This consideration determines the inclination of the straight part of the string to the vertical.

Let θ = the inclination of the axis of the cylinder to AB ;

ϕ = the inclination of the string to the axis;

ω = the angular distance from the lowest generating line of the cylinder of the points where the string leaves the cylinder;

$2a$ = the distance AB ;

$2l$ = the length of the string.

Then, if we project the line AB and the string upon the axis of the cylinder, we have

$$a \cos \theta = l \cos \phi \dots\dots\dots (1).$$

Again, if we project AB and the straight parts of the string produced to meet as above on the plane of either extremity of the cylinder, the line AB would be projected into a line of length $2a \sin \theta$, and the string into two lines, each touching the circular end of the cylinder, and of length $l \sin \psi$: and these lines touching the circular end of the cylinder, they make with each other the angle $\pi - 2\omega$. Hence

$$\cos \omega = \frac{a \sin \theta}{l \sin \phi} \dots\dots\dots (2).$$

Also the produced parts of the string each equals half the part in contact with the cylinder $= (m\pi r + \omega r) \operatorname{cosec} \phi$.

Hence, from the above projected triangle,

$$\begin{aligned}\tan \omega &= \frac{(m\pi r + \omega r) \operatorname{cosec} \phi}{r} \\ &= (m\pi + \omega) \operatorname{cosec} \phi \dots\dots\dots (3).\end{aligned}$$

From (1), (2), and (3), we may determine θ and ϕ ; or the position of the axis of the cylinder and ω : whence the length of the part of the string in contact is known.

Another condition is, that the centre of the cylinder must be symmetrically situated with respect to A and B .

1851.

1. A right cylinder upon an elliptic base (the semiaxes of which are a and b) rests with its axis horizontal between two smooth planes inclined at right angles to each other: determine the position of equilibrium, (1) when the inclination of one of the planes is greater than $\tan^{-1} \frac{a}{b}$, (2) when the inclination of both planes is less than $\tan^{-1} \frac{a}{b}$.

Since the locus of intersection of tangents to an ellipse at right angles to each other is a circle, the locus of the centre of gravity of the cylinder, as the cylinder is turned about in a vertical plane, is a circular arc; and the centre of gravity is at the extremities of this arc when the axes of the cylinder are parallel to the planes. Also these extremities are the lowest points of the arc when the inclination of both the planes is less than $\tan^{-1} \frac{a}{b}$; but if one of them be greater than $\tan^{-1} \frac{a}{b}$, one extremity is the highest point of the arc and the other the lowest: hence, in this case, the position of equilibrium is that in which the major axis is parallel to the plane whose inclination is least; and in the former case there are two positions of equilibrium, viz. when each axis of the cylinder is parallel to either plane.

2. A, B, C , are three rough points in a vertical plane; P, Q, R , are the greatest weights which can be severally supported by a weight W , when connected with it by strings passing over A, B, C , over A, B , and over B, C , respectively: shew that the coefficient of friction at $B = \frac{1}{\pi} \log_{\varepsilon} \frac{QR}{PW}$.

We may consider each of the rough points A, B, C , as cylinders of indefinitely small radius: hence, by a known theorem relating to strings passing over rough surfaces, if θ be the angle through which the string is bent at any of the points whose coefficient of friction is μ , and T_1, T_2 be the tensions of the strings on the two sides of the point, if all possible friction is being exerted, we have

$$T_1 = \varepsilon^{\mu\theta} T_2.$$

Let μ_A, μ_B, μ_C , be the friction at A, B , and C ; α, γ , the inclinations to the horizon of BC and AB respectively: then, by the question,

$$P = \varepsilon^{\mu_A (\frac{1}{2}\pi - \gamma)} \cdot \varepsilon^{\mu_B (\gamma - \alpha)} \cdot \varepsilon^{\mu_C (\frac{1}{2}\pi + \alpha)} \cdot W \dots\dots\dots (1),$$

$$Q = \varepsilon^{\mu_A (\frac{1}{2}\pi - \gamma)} \cdot \varepsilon^{\mu_B (\frac{1}{2}\pi + \gamma)} \cdot W \dots\dots\dots (2),$$

$$R = \varepsilon^{\mu_B (\frac{1}{2}\pi - \alpha)} \cdot \varepsilon^{\mu_C (\frac{1}{2}\pi + \alpha)} \cdot W \dots\dots\dots (3);$$

$\therefore (2) \times (3) \div (1)$ gives

$$\frac{QR}{P} = \varepsilon^{\mu_B \pi} \cdot W;$$

$$\therefore \mu_B = \frac{1}{\pi} \log_{\varepsilon} \frac{QR}{PW}.$$

DYNAMICS OF A PARTICLE.

1848.

1. If a and na be the respective distances of a satellite and of the Sun from a planet, p and mp the periodic times of the satellite and planet, which are supposed to describe circles round the planet and Sun respectively: shew that the orbit of the satellite will always be concave towards the Sun, provided n be greater than m^2 .

Let the angular velocities of the satellite and planet respectively in their orbits be called ω and $m\omega$; then it is plain that the rectangular coordinates of the satellite referred to the Sun as origin and axes rightly chosen, are

$$x = na \cos \omega t + a \cos m\omega t,$$

$$y = na \sin \omega t + a \sin m\omega t.$$

Now, if the path of the satellite pass at any time t from being concave to convex towards the Sun, we have at that time

$$\frac{d^2y}{dx^2} = 0, \quad \text{or} \quad \frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{dy}{dt} \frac{d^2x}{dt^2} = 0;$$

$$\therefore (n \sin \omega t + m \sin m\omega t) (n \sin \omega t + m^2 \sin m\omega t) \\ + (n \cos \omega t + m \cos m\omega t) (n \cos \omega t + m^2 \cos m\omega t) = 0;$$

$$\therefore n^2 + m^3 + mn(m+1) \cos(m-1)\omega t = 0.$$

In order that this equation may not give a possible value of t , we must have

$$n^2 + m^3 > mn(m+1);$$

$$\therefore n^2 - m(m+1)n > -m^3,$$

$$\text{or} \left\{ n - \frac{m(m+1)}{2} \right\}^2 > \frac{m^2(m-1)^2}{2},$$

$$\text{or } n > m^2,$$

which is therefore the condition to be fulfilled, in order that the path of the satellite may be always concave towards the Sun.*

2. A body of given elasticity is projected with a given velocity, and rebounds n times at a horizontal plane passing through the point of projection: determine the direction of projection, so that the angle between the direction of projection and the direction of the ball immediately after the last impact may be the greatest possible.

Let $\alpha, \alpha_1, \alpha_2 \dots \alpha_n$, be the angles of the first projection, and after the successive impacts;

$$\therefore \tan \alpha_n = e \tan \alpha_{n-1} = e^2 \tan \alpha_{n-2} = \dots = e^n \tan \alpha,$$

if e is the modulus of elasticity;

$$\therefore \tan(\alpha - \alpha_n) = \frac{(1 - e^n) \tan \alpha}{1 + e^n \tan^2 \alpha}:$$

we have to determine α , so that this shall be a maximum.

Taking the logarithmic differential of this expression with respect to $\tan \alpha$, we have

$$\frac{1}{\tan \alpha} - \frac{2e^n \tan \alpha}{1 + e^n \tan^2 \alpha} = 0;$$

$$\therefore 1 - e^n \tan^2 \alpha = 0,$$

$$\text{and } \tan \alpha = \frac{1}{e^{\frac{1}{2n}}}.$$

3. If a body be projected with a given velocity about a centre of force which $\propto \frac{1}{(\text{dist.})^2}$, shew that the axis-minor of

* Since the above condition assigns an inferior limit to the value of m (n remaining constant), it manifestly precludes the possibility of a motion of the satellite about the Sun in a direction opposite to that of the planet *i.e.* a retrograde motion as seen from the Sun, which would clearly require m to be greater than when its path is merely alternately concave and convex and not looped.

the orbit described will vary as the perpendicular from the centre of force upon the direction of projection; and determine the locus of the centre of the orbit described.

Let r be the distance, α the angle of projection:

then $b^2 = SY.HZ$, the product of the perpendiculars from the foci on the direction of projection,

$$= SP \sin \alpha . HP \sin \alpha ,$$

$$= r (2a - r) \sin^2 \alpha .$$

And a is constant since the velocity of projection is so ;

$$\therefore b \propto \sin \alpha ,$$

$$\propto r \sin \alpha ,$$

\propto the perpendicular from S upon the direction of projection.

Also, if ρ , ϕ be the polar coordinates of the centre of the curve referred to the centre of force as pole, and initial radius vector as prime radius, $\rho = ae$, $\phi =$ angular distance of the apse, $c =$ distance of projection,

$$\frac{1}{c} = \frac{1}{a} \frac{1 - e \cos \phi}{1 - e^2} ,$$

$$= \frac{1}{a} \frac{1 - \frac{\rho}{a} \cos \phi}{1 - \frac{\rho^2}{a^2}} ,$$

$$\text{or } \rho^2 - c\rho \cos \phi + ac - a^2 = 0 ,$$

from which equation we see that the locus required is a circle.

4. Two bodies, A , B , when acted on by gravity, are projected from two given points in the same vertical line with the same velocity, and in parallel directions: shew that if A be higher than B , a pair of tangents drawn to B 's path from any point of A 's path, will intercept arcs described by B in equal times.

For if we join the two points of contact, the chord so formed will be an ordinate to the vertical diameter through the point in which the tangents meet: let $2y$ be this double ordinate; then, if h be the height of A above B , and the equation to B 's path referred to this diameter be

$$\eta^2 = l'\xi,$$

$$\text{we have } y^2 = l'h.$$

Also, if l be latus-rectum of the parabola, and α the inclination of the ordinates of the above diameter to it, the horizontal distance between the points of contact

$$= 2y \sin \alpha,$$

$$= 2l'^{\frac{1}{2}} \sin \alpha \cdot h^{\frac{1}{2}},$$

$$= 2l^{\frac{1}{2}} h^{\frac{1}{2}},$$

is constant.

Therefore also the time of passage between the points is constant.

5. A body is acted on by a force $= \frac{\mu}{(\text{dist.})^2}$ tending to a fixed centre S : shew that in general there will be two directions, differently inclined to AS , in which the body may be projected from a given point A , with a given velocity v , so as to pass through another given point B .

Prove also that if t, t' be the times of moving from A to B in the two cases, either $t = t'$ or $t + t' = 2\pi\mu \left(\frac{2\mu}{SA} - v^2 \right)^{-\frac{3}{2}}$.

Let the body be projected from A (fig. 95) in a direction making an angle α with the distance, so as to pass through B :

$$SA = a, \quad SB = b, \quad \angle ASB = \beta.$$

The equation to the orbit is

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{v^2 a^2 \sin^2 \alpha} = 0,$$

$$\text{or } u = \frac{\mu}{v^2 a^2 \sin^2 \alpha} \{1 - e \cos(\theta - \gamma)\} \dots \dots \dots (1),$$

$$\frac{du}{d\theta} = \frac{\mu}{v^2 a^2 \sin^2 \alpha} \cdot e \sin(\theta - \gamma).$$

Now, when $\theta = 0$, $u = \frac{1}{a}$,

$$\frac{du}{d\theta} = -\frac{1}{a} \cot \alpha;$$

$$\therefore \frac{1}{a} = \frac{\mu}{v^2 a^2 \sin^2 \alpha} (1 - e \cos \gamma); \quad \frac{1}{a} \cot \alpha = \frac{\mu}{v^2 a^2 \sin^2 \alpha} e \sin \gamma.$$

Eliminating $e \cos \gamma$ and $e \sin \gamma$ from equation (1), we have

$$u = \frac{\mu}{v^2 a^2 \sin^2 \alpha} - \left(\frac{\mu}{v^2 a^2 \sin^2 \alpha} - \frac{1}{a} \right) \cos \theta - \frac{1}{a} \cot \alpha \sin \theta;$$

but when $\theta = \beta$, $u = \frac{1}{b}$;

$$\therefore \frac{1}{b} = \frac{\mu}{v^2 a^2} (1 + \cot^2 \alpha) - \left\{ \frac{\mu}{v^2 a^2} (1 + \cot^2 \alpha) - \frac{1}{a} \right\} \cos \beta - \frac{1}{a} \cot \alpha \sin \beta,$$

a quadratic equation, from which the two values of $\cot \alpha$ can be determined; which proves that there are in general two different directions of projection.

Now (Hymers' *Ast.*, Art. 326) the time from A to B can be determined in terms of the focal distances SA , SB , the chord AB , and the axis-major; and the velocity being given, the axis-major is independent of the direction of projection: hence SA , AB , BS , and the major-axes of the two orbits, are the same. Therefore the periodic time in the two orbits is the same; and also the time from A to B .

If t , t' be the times of describing AB , and the bodies be projected so as both to describe the angle ASB , $t = t'$. But if one describes the angle ASB , and the other $2\pi - ASB$, $t + t'$ equals the periodic time in the conic section $= \frac{2\pi A^{\frac{3}{2}}}{\sqrt{\mu}}$, where

A equals the semiaxis-major $= \frac{\mu SP}{2\mu - v^2 SP}$, for

$$v^2 = \frac{\mu}{SP^2} \cdot \frac{SP(2A - SP)}{A};$$

$$\therefore t + t' = 2\pi\mu \left(\frac{2\mu}{SP} - v^2 \right)^{-\frac{3}{2}}.*$$

* For this solution we are indebted to Mr. Gaskin.

6. Force varying as $\frac{1}{(\text{dist.})^2}$, shew that, when the latus-rectum is given, an angle $= 2 \tan^{-1} 5^{\frac{1}{4}}$, measured from the nearer apse, will be described very nearly in the same time, whether the body moves in an elliptic or an hyperbolic orbit, whose eccentricities are $1 - \alpha$ and $1 + \alpha$ respectively, α being small.

We have

$$\frac{d\theta}{dt} = \frac{h}{r^2},$$

$$\text{and } r = l \{1 + (1 \mp \alpha) \cos \theta\}^{-1},$$

in the ellipse and hyperbola respectively, where l is the common latus-rectum: also $h^2 = \mu l$ is the same in both cases; hence

$$\begin{aligned} \frac{dt}{d\theta} &= \frac{l^2}{h} \{1 + (1 \mp \alpha) \cos \theta\}^{-2} \\ &= \frac{l^2}{h} (2 \cos^2 \tfrac{1}{2} \theta \mp \alpha \cos \theta)^{-2} \\ &= \frac{l^2}{h} \frac{\sec^4 \tfrac{1}{2} \theta}{4} \left(1 \mp \alpha \frac{\cos \theta}{\cos^2 \tfrac{1}{2} \theta}\right)^{-2} \\ &= \frac{l^2}{4h} \sec^4 \tfrac{1}{2} \theta \left(1 \pm 2\alpha \frac{\cos \theta}{\cos^2 \tfrac{1}{2} \theta}\right) \text{ very nearly;} \\ \therefore t &= \frac{l^2}{2h} \int (\sec^2 \tfrac{1}{2} \theta) \{1 \pm 2\alpha (\sec^2 \tfrac{1}{4} \theta - 2)\} d \tan \tfrac{1}{2} \theta \\ &= \frac{l^2}{2h} \int \{1 + \tan^2 \tfrac{1}{2} \theta \pm 2\alpha (\tan^4 \tfrac{1}{2} \theta - 1)\} d \tan \tfrac{1}{2} \theta. \end{aligned}$$

Hence, if T be the time of describing an angle β from the nearer apse,

$$T = \frac{l^2}{2h} \left\{ \tan \tfrac{1}{2} \beta + \tfrac{1}{3} \tan^3 \tfrac{1}{2} \beta \pm 2\alpha \left(\tfrac{1}{5} \tan^5 \tfrac{1}{2} \beta - \tan \tfrac{1}{2} \beta \right) \right\}.$$

Hence the difference of times of describing this arc in the two cases

$$= \frac{2\alpha l^2}{h} \left(\tfrac{1}{5} \tan^5 \tfrac{1}{2} \beta - \tan \tfrac{1}{2} \beta \right),$$

which vanishes if $\beta = 2 \tan^{-1} 5^{\frac{1}{4}}$, and the proposition is true.

1849.

1. If the equation for determining the apsidal distances in a central orbit contain the factor $(u-a)^2$, shew that a will be a root of the equation

$$\phi(u) - h^2 u^3 = 0,$$

where $\phi(u)$ is the central force.

The differential equation of the orbit will be

$$\frac{d^2 u}{d\theta^2} + u = \frac{\phi(u)}{h^2 u^2} \dots\dots\dots (1).$$

Multiply by $2 \frac{du}{d\theta}$, and integrate;

$$\therefore \left(\frac{du}{d\theta} \right)^2 = 2 \int \frac{\phi(u) \cdot du}{h^2 u^2} - u^2.$$

The general condition for an apse is, that $\frac{du}{d\theta} = 0$; and therefore the equation for determining the apsidal distances is

$$2 \int \frac{\phi(u) du}{h^2 u^2} - u^2 = 0.$$

If this equation contain the factor $(u-a)^2$, let us suppose that

$$2 \int \frac{\phi(u) \cdot du}{h^2 u^2} - u^2 = \overline{f(u)}^2 \cdot (u-a)^2;$$

$$\text{then } \frac{du}{d\theta} = (u-a) \cdot f(u),$$

$$\begin{aligned} \text{and } \frac{d^2 u}{d\theta^2} &= \frac{du}{d\theta} \cdot \frac{d}{du} \cdot \left(\frac{du}{d\theta} \right) \\ &= f(u) \{ f(u) + f'(u) \cdot (u-a) \} (u-a); \end{aligned}$$

hence $u = a$ is a root of the equation

$$\frac{d^2 u}{d\theta^2} = 0;$$

that is, a root of the equation

$$\phi(u) - h^2 u^3 = 0.$$

2. A body moves from rest at a distance a towards a centre of force, the force varying inversely as the distance: shew that the time of describing the space between βa and $\beta^n a$ will be a maximum if $\beta = \frac{1}{n^{\frac{1}{2(n-1)}}}$.

We have here

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x};$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = 2\mu \log \frac{a}{x},$$

since $x = a$, when $\frac{dx}{dt} = 0$;

$$\begin{aligned} \therefore t &= \frac{1}{(2\mu)^{\frac{1}{2}}} \int \frac{dx}{\left(\log \frac{a}{x}\right)^{\frac{1}{2}}} \\ &= \frac{1}{(2\mu)^{\frac{1}{2}}} \{F(x) + C\} \text{ suppose.} \end{aligned}$$

Now, let T be the time of describing the space between βa and $\beta^n a$; then

$$T = \frac{1}{(2\mu)^{\frac{1}{2}}} \{F(\beta^n a) - F(\beta a)\}.$$

In order that this may be a maximum, we must have

$$\frac{dT}{d\beta} = 0;$$

$$\therefore n\beta^{n-1} F'(\beta^n a) - F'(\beta a) = 0.$$

$$\text{But } F'(x) = \frac{1}{\left(\log \frac{a}{x}\right)^{\frac{1}{2}}};$$

therefore the above condition becomes

$$\frac{n\beta^{n-1}}{\left(\log \frac{1}{\beta^n}\right)^{\frac{1}{2}}} - \frac{1}{\left(\log \frac{1}{\beta}\right)^{\frac{1}{2}}} = 0,$$

$$\text{or } n^{\frac{1}{2}}\beta^{n-1} - 1 = 0;$$

$$\therefore \beta = \frac{1}{n^{\frac{1}{2(n-1)}}},$$

the required expression.

3. A particle is attached to the extremity of a fine string, which is partially wound round a cylinder of diameter c ; if the unwound portion of the string be kept stretched, and the particle be projected perpendicularly to its length with a velocity V , prove that the string will be wound up after the lapse of the time $\frac{l^2}{Vc}$, where l is the length of string unwound at the time of projection.

Let r be the length of the string unwound at the time t after projection, $\frac{c}{2}\theta$ the arc which has become covered with string in that time: then, since the only force on the particle, viz. the tension of the string, is always perpendicular to its instantaneous direction of motion, the velocity of the particle is uniform;

$$\therefore r \frac{d\theta}{dt} = \text{a constant:}$$

and at the time of projection

$$r \frac{d\theta}{dt} = V;$$

$$\therefore r \frac{d\theta}{dt} = V.$$

$$\text{Also, } r = l - \frac{c}{2}\theta;$$

$$\therefore \left(l - \frac{c}{2}\theta\right) \frac{d\theta}{dt} = V,$$

$$\text{and } \frac{1}{2} \left(l - \frac{c}{2}\theta\right)^2 = C - \frac{c}{2} Vt;$$

and when $t = 0$, $\theta = 0$; $\therefore C = \frac{1}{2}l^2$;

$$\therefore \left(l - \frac{c}{2}\theta\right)^2 = l^2 - c Vt.$$

Let T be the time when the string is all wound up;

$$\therefore t = T, \text{ when } l - \frac{c}{2} \theta = 0;$$

$$\therefore T = \frac{l^2}{Vc}.$$

4. A particle describes an ellipse about a centre of force in the focus S (fig. 96); about S as centre a circle is described, which is cut by the radius vector SP in the point Q ; from Q a line is drawn perpendicular to the direction of the particle's motion, which meets the major-axis in R : prove that R is constant in position, and that QR is proportional to the particle's velocity throughout the motion.

From P draw the normal PG , and from S the perpendicular SY upon the tangent; also draw GL perpendicular to SP , PL is half the latus-rectum.

Now QR is parallel to PG ;

$$\begin{aligned} \therefore SR &= \frac{SG}{SP} \cdot SQ \\ &= e \cdot SQ, \end{aligned}$$

by the property of the ellipse; therefore SR is constant.

$$\begin{aligned} \text{Again, } QR &= \frac{PG}{SP} \cdot SQ = \frac{PL \sec LPG}{SP} \cdot SQ \\ &= \frac{PL \cdot SQ}{SP \cos PSY} = \frac{PL \cdot SQ}{SY} \propto \frac{1}{SY}; \end{aligned}$$

and velocity $\propto \frac{1}{SY}$;

$$\therefore QR \propto \text{velocity}.$$

Q. E. D.

1850.

1. A heavy particle is fastened by two equal strings of given length to two points in a horizontal line, and then whirled round in a vertical plane; the velocity is such that, if one of the strings break when the particle is either at its lowest point or half-way between its highest and lowest points, the particle

will still continue to describe a circle: find the least distance between the point to which the strings are fastened that this may be possible.

Let l be the length of either string, θ be the inclination of the strings to the horizon when the distance between the points is the least possible. Let V be the corresponding velocity of the particle at its lowest point before the string breaks; then $(V^2 - 2gl \sin \theta)^{\frac{1}{2}}$ will be its velocity when the strings are horizontal.

Now, if one of the strings break when the particle is at its lowest point, it will proceed in a horizontal circle about the vertical line through the point of support of the unbroken string, if the velocity be such as to produce a centrifugal force just sufficient to keep the string at the same inclination to the horizon, or, resolving the forces perpendicular to the length of the string, if

$$\begin{aligned} \frac{V^2}{l \cos \theta} \cdot \sin \theta &= g \cos \theta, \\ \text{or } V^2 &= gl \frac{\cos^2 \theta}{\sin \theta} \dots\dots\dots (1). \end{aligned}$$

If one of the strings break when the particle is half-way between its lowest and highest points, it will proceed to describe a vertical circle about the point of support of the other string, provided the velocity $(V^2 - 2gl \sin \theta)^{\frac{1}{2}}$ be great enough to carry the particle over the highest point of the circle, *i.e.* if

$$V^2 - 2gl \sin \theta > 3gl.$$

Now, θ is supposed to have received its greatest possible value, and therefore, from (1), V its least possible value; hence

$$\begin{aligned} V^2 - 2gl \sin \theta &= 3gl, \\ \text{or } gl \frac{\cos^2 \theta}{\sin \theta} - 2gl \sin \theta &= 3gl; \\ \therefore 1 - \sin^2 \theta - 2 \sin^2 \theta &= 3 \sin \theta, \\ \text{or } \sin^2 \theta + \sin \theta &= \frac{1}{3}, \\ \text{and } \sin \theta &= -\frac{1}{2} + \left(\frac{1}{4} + \frac{1}{3}\right)^{\frac{1}{2}} \\ &= \frac{\left(\frac{7}{3}\right)^{\frac{1}{2}} - 1}{2}. \end{aligned}$$

And the least distance between the points of support

$$\begin{aligned}
 &= 2l \cos \theta \\
 &= [4 - \{(\frac{7}{3})^{\frac{1}{2}} - 1\}^2]^{\frac{1}{2}} l \\
 &= (\frac{2}{3})^{\frac{1}{2}} \{1 + (21)^{\frac{1}{2}}\}^{\frac{1}{2}} l.
 \end{aligned}$$

2. If P be the perimeter of a closed curve described about a centre of force, τ the time of a revolution, h twice the area described in a unit of time, and ρ the radius of curvature at the time t , prove that $P = h \int_0^\tau \frac{dt}{\rho}$.

We have

$$\begin{aligned}
 P &= \int_0^\tau v dt \\
 &= h \int_0^\tau \frac{dt}{p} \\
 &= \int_0^{2\pi} \frac{r^2 d\theta}{p} \\
 &= \int \frac{r \frac{d\theta}{dr} \cdot r dr}{p} \left. \vphantom{\int} \right\} \text{between proper limits.} \\
 &= \int \frac{r dr}{(r^2 - p^2)^{\frac{1}{2}}}
 \end{aligned}$$

Now let $(r^2 - p^2)^{\frac{1}{2}} = f(\theta)$;

$$\therefore \int \frac{r dr - p dp}{(r^2 - p^2)^{\frac{1}{2}}} \text{ between the above limits}$$

$$\begin{aligned}
 &= \int_0^{2\pi} f'(\theta) d\theta \\
 &= f(2\pi) - f(0) = 0;
 \end{aligned}$$

$$\begin{aligned}
 \therefore P &= \int \frac{p dp}{(r^2 - p^2)^{\frac{1}{2}}} \\
 &= \int \frac{pr dr}{(r^2 - p^2)^{\frac{1}{2}} r \frac{dr}{dp}}
 \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{r^2 \frac{d\theta}{dr} dr}{\rho} \\
 &= \int_0^{2\pi} \frac{r^2 d\theta}{\rho} \\
 &= h \int_0^{\tau} \frac{dt}{\rho}.
 \end{aligned}$$

3. If any number of bodies be projected from a given point with the same velocity in one plane, and describe ellipses round a central force which varies inversely as the square of the distance; find the law of force tending to the same centre, under the action of which a body will describe the curve which is the locus of the centres of the different ellipses.

Let μ be the absolute force, V the velocity, and c the distance of projection. Then, if a be the axis-major,

$$\frac{1}{a} = \frac{2}{c} - \frac{V^2}{\mu} \dots\dots\dots (1).$$

Also the equation to the orbit is

$$\frac{1}{r} = \frac{1}{a} \frac{1 - e \cos(\theta - \alpha)}{1 - e^2};$$

and our object is to find the relation between e and α ; for if ρ, ϕ be the polar coordinates of the centre of the ellipse,

$$\rho = ae, \quad \phi = \alpha.$$

Now, when $\theta = 0, r = c$;

$$\therefore \frac{1}{c} = \frac{1}{a} \frac{1 - e \cos \alpha}{1 - e^2},$$

$$\text{or } \frac{1}{c} = \frac{1}{a} \frac{1 - \frac{\rho}{a} \cos \phi}{1 - \frac{\rho^2}{a^2}}.$$

And from (1) a is constant; hence this equation shews that the locus required is a circle; we may put it in the form

$$\frac{\rho^2}{c} - \rho \cos \phi = a^2 \left(\frac{1}{c} - \frac{1}{a} \right) \dots\dots\dots (2).$$

Now, by *Newton*, Sect. 11. Prop. 7, if F be the force in the circle,

$$F \propto \frac{1}{SP^2} \cdot \frac{1}{PV^3}.$$

And from equation (2),

$$SP \cdot SV = a^2 c \left(\frac{1}{a} - \frac{1}{c} \right);$$

$$\therefore SV = a^2 c \left(\frac{1}{a} - \frac{1}{c} \right) \frac{1}{\rho},$$

$$\text{and } PV = \rho + a^2 c \left(\frac{1}{a} - \frac{1}{c} \right) \frac{1}{\rho};$$

$$\therefore F \propto \frac{1}{\rho^2} \frac{1}{\left\{ \rho + a^2 c \left(\frac{1}{a} - \frac{1}{c} \right) \frac{1}{\rho} \right\}^3},$$

$$\propto \frac{\rho}{\left\{ \rho^2 + \frac{1 - \frac{\mu}{V^2 c}}{\left(\frac{2}{c} - \frac{1}{\mu} \right)^2} \right\}^3}.$$

1851.

1. If a body be acted on by a vertical force so as to describe the common catenary, shew that the force and velocity at any point will vary as the distance of that point from the directrix.

The equation to the catenary from the directrix, as axis of x , which we suppose horizontal, is

$$y = \frac{1}{2}c \left(e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right),$$

and the force is wholly vertical;

$$\therefore \frac{d^2x}{dt^2} = 0,$$

$$\text{and } \frac{dx}{dt} = \text{constant} = V \text{ suppose.}$$

$$\text{Also } \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{1}{2} V (e^{\frac{x}{c}} - e^{-\frac{x}{c}});$$

\therefore if v = whole velocity,

$$v^2 = V^2 + \frac{1}{4} V^2 (e^{\frac{x}{c}} - e^{-\frac{x}{c}})^2$$

$$= \frac{1}{4} V^2 (e^{\frac{x}{c}} + e^{-\frac{x}{c}})^2,$$

$$\text{and } v = \frac{1}{2} V (e^{\frac{x}{c}} + e^{-\frac{x}{c}})$$

$$= \frac{V}{c} y$$

$$\propto y;$$

or the velocity at any point varies as the distance of that point from the directrix.

$$\text{Again, } \frac{d^2y}{dt^2} = \frac{d}{dx} \left(\frac{dy}{dt} \right) \frac{dx}{dt}$$

$$= V \frac{d}{dx} \left\{ \frac{1}{2} V (e^{\frac{x}{c}} - e^{-\frac{x}{c}}) \right\}$$

$$= \frac{V^2}{2c} (e^{\frac{x}{c}} + e^{-\frac{x}{c}})$$

$$= \frac{V^2}{c^2} y$$

$$\propto y;$$

or the force at any point varies as the distance of that point from the directrix.

2. Force varying inversely as the square of the distance, a body is projected from a given point in a direction making an angle of 45° with the distance, and with a velocity = n times the velocity in a circle at the same distance: shew that the direction of the major-axis will be unaltered when the angle of projection is increased to $\cot^{-1}(1 - n^2)$.

The general expressions for the elements of the orbit in terms of the distance (c), the velocity (V), and the angle (β) of projection, are

$$\frac{1}{a} = \frac{2}{c} - \frac{V^2}{\mu},$$

$$e \cos \alpha = \frac{V^2 c \sin^2 \beta}{\mu} - 1, \quad \alpha \text{ the apsidal angle,}$$

$$\text{and } e \sin \alpha = \frac{V^2 c \sin \beta \cos \beta}{\mu};$$

$$\therefore \cot \alpha = \tan \beta - \frac{\mu}{V^2 c \sin \beta \cos \beta}.$$

Now, in the present case,

$$V^2 = n^2 \text{ (velocity in a circle at distance } c)$$

$$= n^2 \frac{\mu}{c^2} c = \frac{n^2 \mu}{c};$$

$$\therefore \cot \alpha = \tan \beta - \frac{1}{n^2 \sin \beta \cos \beta}$$

$$= \tan \beta - \frac{1 + \tan^2 \beta}{n^2 \tan \beta};$$

$$\therefore \text{ if } \beta = 45^\circ, \quad \cot \alpha = 1 - \frac{2}{n^2},$$

$$\text{and if } \beta = \cot^{-1}(1 - n^2),$$

$$\cot \alpha = \frac{1}{1 - n^2} - \frac{1 + (1 - n^2)^2}{n^2 (1 - n^2)} = \frac{n^2 - 1 - (1 - n^2)^2}{n^2 (1 - n^2)} = -\frac{1 + 1 - n^2}{n^2} = 1 - \frac{2}{n^2};$$

hence the apsidal angle, and therefore the direction of the axis-major, is the same in the two cases.

3. A body describes a parabola under the action of two equal forces, one tending to the focus and varying inversely as the distance, the other parallel to the axis: find the velocity at any point and the time of moving between the vertex and the extremity of the latus-rectum.

The resultant of the two equal forces will bisect the angle between them, and therefore be normal to the parabola: hence

the velocity is constant; and if ρ be the radius of curvature at P ,

$$\begin{aligned}\frac{v^2}{\rho} &= \text{resultant of the two forces} \\ &= 2 \frac{\mu}{SP} \sin SPY \\ &= \frac{2\mu \cdot SY}{SP^2};\end{aligned}$$

$$\text{but } \rho = \frac{2SP^2}{SY},$$

$$\therefore v^2 = 4\mu,$$

and $v = 2\mu^{\frac{1}{2}}$, the required value.

Again, if S be the length of the arc from the vertex to the extremity of the latus-rectum, the time (T) of moving over it

$$T = 2\mu^{\frac{1}{2}}S.$$

Now,

$$S = \int_0^l \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}} dx,$$

$$\text{where } y^2 = 4lx,$$

$$\text{and } \therefore \frac{dy}{dx} = \left(\frac{l}{x} \right)^{\frac{1}{2}};$$

$$\begin{aligned}\therefore S &= \int_0^l \left(1 + \frac{l}{x} \right)^{\frac{1}{2}} dx \\ &= \int_0^l \frac{l+x}{(lx+x^2)^{\frac{1}{2}}} dx \\ &= \int_0^l \frac{\frac{l}{2} + \left(\frac{l}{2} + x \right)}{\left\{ \frac{3l^2}{4} + \left(\frac{l}{2} + x \right)^2 \right\}^{\frac{1}{2}}} dx \\ &= \frac{1}{2}l \log \frac{\frac{l}{2} + x + (lx+x^2)^{\frac{1}{2}}}{c} + (lx+x^2)^{\frac{1}{2}} \\ &= \frac{1}{2}l \log (3+2 \cdot 2^{\frac{1}{2}}) + 2^{\frac{1}{2}} \cdot l \text{ (between the limits)} \\ &= \{ \log (1+2^{\frac{1}{2}}) + 2^{\frac{1}{2}} \} l; \\ \therefore T &= 2\mu^{\frac{1}{2}} \{ \log (1+2^{\frac{1}{2}}) + 2^{\frac{1}{2}} \} l.\end{aligned}$$

4. If the product of the velocities at two points P, Q of the parabolic path of a body acted on by gravity be constant, shew that the locus of the pole of PQ is a circle having the focus of the parabola for its centre.

Let α, β be the angles between the axis of the parabola and SP, SQ respectively; then the equations to the tangents at P, Q referred to S as pole are

$$\frac{1}{r} = \frac{2}{l} \{ \cos \theta + \cos(\theta - \alpha) \},$$

$$\frac{1}{r} = \frac{2}{l} \{ \cos \theta + \cos(\theta - \beta) \}.$$

Hence, at the pole of PQ ,

$$\cos(\theta - \alpha) = \cos(\theta - \beta),$$

and α does not equal β ; therefore

$$\theta = \frac{\alpha + \beta}{2},$$

and at the pole

$$\begin{aligned} \frac{1}{r} &= \frac{2}{l} \left(\cos \frac{\alpha + \beta}{2} + \cos \frac{\alpha - \beta}{2} \right) \\ &= \frac{4}{l} \cos \frac{\alpha}{2} \cos \frac{\beta}{2}. \end{aligned}$$

$$\text{Now } SP = \frac{l}{4} \sec^2 \frac{\alpha}{2},$$

$$SQ = \frac{l}{4} \sec^2 \frac{\beta}{2},$$

and velocity² at $P = 2g.SP$,

..... $Q = 2g.SQ$;

$\therefore SP.SQ = \text{constant}$ (by the question),

or $\sec^2 \frac{1}{2}\alpha \sec^2 \frac{1}{2}\beta = \text{constant}$;

therefore the value of r at the pole is constant, or the locus of that point is a circle about the focus as centre.

5. Force varying as the distance, let P, Q (fig. 97) be two points in the orbit described by a body round a given centre of force C , and let PT (the tangent at the point P) meet CQ produced in T ; join PQ and draw TA parallel to PQ meeting CP produced in A ; draw QA meeting PT in U , and CU meeting PQ in V ; in CU take CR a mean proportional between CV and CU , then the body will pass through the point R , and the time of moving from P to R will be half the time of moving from P to Q .

Let $CP = a$, $CQ = b$, then the equation to the ellipse referred to CP, CQ as axes will be

$$\frac{x^2}{a^2} + \frac{2xy}{c^2} + \frac{y^2}{b^2} = 1 \dots\dots\dots(1).$$

Let $CA = a'$, $CT = b'$; the equation to PT will be

$$\frac{x}{a} + \frac{y}{b'} = 1 \dots\dots\dots(2).$$

To express the condition in order that this may touch the ellipse, take $(1) - (2)^2$; therefore

$$2 \left(\frac{1}{c^2} - \frac{1}{ab'} \right) xy + \left(\frac{1}{b^2} - \frac{1}{b'^2} \right) y^2 = 0,$$

an equation to be satisfied only by the coordinates $x = a$, $y = 0$; therefore

$$\begin{aligned} \frac{1}{c^2} - \frac{1}{ab'} &= 0, \\ \text{or } b' &= \frac{c^2}{a}. \end{aligned}$$

Also, since AT is parallel to PQ ,

$$a' = \frac{a}{b} \cdot b' = \frac{c^2}{b},$$

which is evidently the condition in order that QA may touch the ellipse.

Hence QU, PU both touch the ellipse, and by a known property of the ellipse, R is a point in the curve if $CR^2 = CU \cdot CV$.

Again, let θ_1, ϕ, θ_2 be the angles CP, CR, CQ respectively make with the apsidal distance, then the time in which the particle will reach P from the apse

$$= \frac{1}{\mu^{\frac{1}{2}}} \tan^{-1} \left(\frac{a}{b} \tan \theta_1 \right),$$

where a and b are now the semiaxes.

Let $(h_1, k_1), (h_2, k_2)$ be the coordinates of the points P, Q referred to the axes of the figure, the equations to PU, QU will be

$$\frac{h_1}{a^2} x + \frac{k_1}{b^2} y = 1,$$

$$\text{and } \frac{h_2}{a^2} x + \frac{k_2}{b^2} y = 1,$$

and the equation to CU through their point of intersection

$$\left(\frac{h_1}{a^2} x + \frac{k_1}{b^2} y - 1 \right) + \lambda \left(\frac{h_2}{a^2} x + \frac{k_2}{b^2} y - 1 \right) = 0,$$

and this line passes through the origin; therefore

$$\lambda = -1,$$

and the equation to CU becomes

$$\frac{h_1 - h_2}{a^2} x + \frac{k_1 - k_2}{b^2} y = 0;$$

$$\therefore \tan \phi = - \frac{\frac{h_1 - h_2}{a^2}}{\frac{k_1 - k_2}{b^2}};$$

therefore time from P to R

$$\begin{aligned} &= \frac{1}{\mu^{\frac{1}{2}}} \left\{ \tan^{-1} \left(\frac{a}{b} \tan \phi \right) - \tan^{-1} \left(\frac{a}{b} \tan \theta_1 \right) \right\} \\ &= \frac{1}{\mu^{\frac{1}{2}}} \cdot \frac{a}{b} \frac{\tan \phi - \tan \theta_1}{1 + \frac{a^2}{b^2} \tan \theta_1 \tan \phi} \end{aligned}$$

$$\begin{aligned}
& \frac{\frac{h_1}{a^2} - \frac{h_1 - h_2}{a^2}}{\frac{k_1}{b^2} - \frac{k_1 - k_2}{b^2}} \\
&= \frac{ab}{\mu^{\frac{1}{2}}} \frac{\frac{h_1(h_1 - h_2)}{a^4}}{b^2 + a^2 \frac{k_1(k_1 - k_2)}{b^4}} \\
&= \frac{ab}{\mu^{\frac{1}{2}}} \frac{\frac{k_1 h_2 - h_1 k_2}{a^2 b^2}}{\frac{k_1(k_1 - k_2)}{b^2} + \frac{h_1(h_1 - h_2)}{a^2}} \\
&= \frac{1}{\mu^{\frac{1}{2}} ab} \cdot \frac{k_1 h_2 - h_1 k_2}{1 - \frac{h_1 h_2}{a^2} - \frac{k_1 k_2}{b^2}},
\end{aligned}$$

an expression which only changes sign when the suffixes 1 and 2 are interchanged; it therefore

$$\begin{aligned}
&= \frac{1}{\mu^{\frac{1}{2}}} \left\{ \tan^{-1} \left(\frac{a}{b} \tan \theta_2 \right) - \tan^{-1} \left(\frac{a}{b} \tan \phi \right) \right\} \\
&= \text{time from } R \text{ to } Q.
\end{aligned}$$

Q. E. D.

6. Two bodies A and B revolve in the same conic section round the same centre of force in the focus; shew that if A and B be at opposite extremities of any focal chord, B will appear (to a spectator on A) to move with a constant velocity perpendicular to SP , or with a constant velocity perpendicular to the transverse axis, according as A and B describe the conic section in the same or opposite directions.

Let PT , QT (fig. 98) be the tangents at the points P and Q at the extremity of the focal chord PSQ ; draw SY , SY' perpendiculars from S upon those tangents.

First, suppose the bodies A and B to be moving in the same direction about S , then the relative velocity of A and B perpendicular to PSQ

$$= V_1 \sin SPY + V_2 \sin SQY',$$

if V_1 and V_2 be the velocities of A and B

$$\begin{aligned}
 &= \frac{h}{SY} \cdot \frac{SY}{SP} + \frac{h}{SY'} \cdot \frac{SY'}{SP'} \\
 &= h \left(\frac{1}{SP} + \frac{1}{SP'} \right) \\
 &= \frac{h}{\frac{1}{2} \text{ lat. rectum}} \text{ is constant.}
 \end{aligned}$$

Secondly, suppose A and B to be moving in opposite directions about S , then their relative velocity perpendicular to the transverse axis

$$\begin{aligned}
 &= V_1 \cos PTS - V_2 \cos QT'S \\
 &= \frac{h}{SP \sin SPY} \cdot \frac{1}{e} \cos SPY - \frac{h}{SP' \sin SQY'} \cdot \frac{1}{e} \cos SQY' \\
 &= \frac{h}{e} \left(\frac{1}{SP} \cot SPY - \frac{1}{SP'} \cot SQY' \right) \\
 &= \frac{h}{e} (u \cot \phi - u' \cot \phi') \text{ suppose.}
 \end{aligned}$$

Let $ASP = \theta$, then

$$u = \frac{2}{l} (1 + e \cos \theta) \quad (l \text{ the latus-rectum}),$$

$$\text{and } \cot \phi = -\frac{1}{u} \frac{du}{d\theta},$$

$$\therefore u \cot \phi = -\frac{du}{d\theta} = \frac{2e}{l} \sin \theta,$$

and writing $\pi + \theta$ for θ , and neglecting the change of sign, since $\cot \phi' = \frac{1}{u'} \frac{du'}{d\theta}$ and not $-\frac{1}{u'} \frac{du'}{d\theta}$ as above, we find

$$u' \cot \phi' = \frac{2e}{l} \sin \theta,$$

whence we see that the above expression gives the relative velocity perpendicular to the transverse axis equal to zero.

7. A body m moves with a uniform velocity $v = \frac{\sec \alpha}{2a} \mu^{\frac{1}{2}}$ in a circular tube whose radius is a , and attracts a body m' within the same tube with a force $= \frac{\mu}{(\text{dist.})^3}$: shew that if m and m' be originally situated in the opposite extremities of a diameter and m' at rest, the two bodies will meet one another at the end of the time $\frac{2a\alpha}{b \sin \alpha}$.

Let P, Q (fig. 99) be the positions of m' and m at the time t after the beginning of motion, and let PCA, QCB be their angular distances from their original positions A, B at the extremities of the diameter ACB ; let $PCA = \theta$, $PCQ = \phi$; also $QCB = \frac{v}{a} t$: join PQ ; then, for the motion of m' ,

$$\begin{aligned} a \frac{d^2 \theta}{dt^2} &= \frac{\mu}{PQ^3} \sin CPQ \\ &= \frac{\mu}{(2a)^3} \frac{\cos \frac{1}{2} \phi}{\sin^3 \frac{1}{2} \phi}. \end{aligned}$$

$$\text{Also, } \theta = \pi - \phi - \frac{v}{a} t;$$

$$\therefore \frac{d^2 \theta}{dt^2} = - \frac{d^2 \phi}{dt^2},$$

$$\text{and } 2 \frac{d\phi}{dt} \frac{d^2 \phi}{dt^2} = - \frac{\mu}{4a^4} \frac{\cos \frac{1}{2} \phi}{\sin^3 \frac{1}{2} \phi} \frac{d\phi}{dt};$$

$$\begin{aligned} \therefore \left(\frac{d\phi}{dt} \right)^2 &= \frac{\mu}{4a^4} \left(\frac{1}{\sin^2 \frac{1}{2} \phi} + C \right) \\ &= \frac{v^2 \cos^2 \alpha}{a^2} \left(\frac{1}{\sin^2 \frac{1}{2} \phi} + C \right). \end{aligned}$$

$$\text{Now, when } t = 0, \frac{d\phi}{dt} = - \frac{v}{a}, \phi = \pi;$$

$$\therefore \frac{v^2}{a^2} = \frac{v^2 \cos^2 \alpha}{a^2} (1 + C);$$

$$\therefore C = \sec^2 \alpha - 1,$$

$$\text{and } \left(\frac{d\phi}{dt}\right)^2 = \frac{v^2 \cos^2 \alpha}{a^2} \left(\frac{1}{\sin^2 \frac{1}{2} \phi} + \sec^2 \alpha - 1 \right),$$

$$\begin{aligned} \therefore \frac{dt}{d\phi} &= - \frac{a}{v \cos \alpha} \frac{\sin \frac{1}{2} \phi}{\{1 + \sin^2 \frac{1}{2} \phi (\sec^2 \alpha - 1)\}^{\frac{1}{2}}} \\ &= - \frac{a}{v \cos \alpha} \frac{\sin \frac{1}{2} \phi}{\{\sec^2 \alpha - (\sec^2 \alpha - 1) \cos^2 \frac{1}{2} \phi\}^{\frac{1}{2}}}, \end{aligned}$$

$$\begin{aligned} \text{and } t &= \frac{2a}{v \cos \alpha} \frac{1}{(\sec^2 \alpha - 1)^{\frac{1}{2}}} \sin^{-1} \frac{(\sec^2 \alpha - 1)^{\frac{1}{2}}}{\sec \alpha} \cos \frac{1}{2} \phi + C \\ &= \frac{2a}{v \sin \alpha} \sin^{-1} \sin \alpha \cos \frac{1}{2} \phi + C (= 0). \end{aligned}$$

And if T be the time that elapses before the collision,

$$\begin{aligned} T &= \frac{2a}{v \sin \alpha} \sin^{-1} \sin \alpha \\ &= \frac{2a\alpha}{v \sin \alpha}. \end{aligned}$$

8. A straight rod AB (100) slides between two planes OA , OB , one of which is horizontal and the other vertical: then, if a body acted on by gravity descends from rest from the highest point, down the curve which always touches AB , the time down any arc : the time down the corresponding chord :: twice the arc : the chord.

Let the length of the rod $AB = c$, the equation to the curve which always touches AB , referred to Cx , Cy , the horizontal and vertical axes through its highest point, is

$$(c - x)^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

We will shew that CD is the curve which has the required property. Let x' , y' be the coordinates of the point P . Then

$$\text{time down arc } CP = \int_0^{y'} \frac{ds}{\sqrt{2gy}},$$

$$\dots \text{ chord } CP = \frac{2^{\frac{1}{2}} l}{(gy)^{\frac{1}{2}}}, \quad l \text{ the length of the chord.}$$

Hence the property in the question gives

$$\begin{aligned}
 \frac{\int_0^{v'} \frac{ds}{(2gy)^{\frac{1}{2}}} dy}{\frac{2^{\frac{1}{2}} l}{(gy')^{\frac{1}{2}}}} &= \frac{2 \int_0^{v'} \frac{ds}{dy} dy}{l}; \\
 \therefore \int_0^{v'} \frac{ds}{(gy')^{\frac{1}{2}}} dy &= \frac{4}{(gy')^{\frac{1}{2}}} \int_0^{v'} \frac{ds}{dy} dy; \\
 \therefore \frac{1}{(gy')^{\frac{1}{2}}} \frac{ds'}{dy'} &= \frac{4}{(gy')^{\frac{1}{2}}} \frac{ds'}{dy'} - \frac{2}{g^{\frac{1}{2}} y^{\frac{3}{2}}} \int_0^{v'} \frac{ds}{dy} dy; \\
 \therefore \frac{3}{(gy')^{\frac{1}{2}}} \frac{ds'}{dy'} &= \frac{2}{(gy')^{\frac{1}{2}}} \cdot \frac{1}{y'} \int_0^{v'} \frac{ds}{dy} dy; \\
 \therefore \frac{3}{2} y' \frac{ds'}{dy'} &= \int_0^{v'} \frac{ds}{dy} dy; \\
 \therefore \frac{3}{2} y' \frac{d^2 s'}{dy'^2} + \frac{3}{2} \frac{ds'}{dy'} &= \frac{ds'}{dy'};
 \end{aligned}$$

or dropping the accents,

$$\begin{aligned}
 \frac{ds}{dy} + 3y \frac{d^2 s}{dy^2} &= 0, \\
 \text{or } \frac{1}{y} + 3 \frac{\frac{d^2 s}{dy^2}}{\frac{ds}{dy}} &= 0, \\
 \log y + 3 \log \frac{ds}{dy} &= \log c; \\
 \therefore \frac{ds}{dy} &= \left(\frac{c}{y} \right)^{\frac{1}{3}}, \\
 \text{or } 1 + \left(\frac{dx}{dy} \right)^2 &= \left(\frac{c}{y} \right)^{\frac{2}{3}}, \\
 \frac{dx}{dy} &= \frac{(c^{\frac{2}{3}} - y^{\frac{2}{3}})^{\frac{1}{2}}}{y^{\frac{1}{3}}};
 \end{aligned}$$

$$\therefore x + C = -(c^{\frac{2}{3}} - y^{\frac{2}{3}})^{\frac{3}{2}},$$

and the curve passes through the origin;

$$\therefore C = -c,$$

$$\text{and } (x-c)^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$$

is its equation, shewing that the curve which has the required property is the curve generated as CD is.

RIGID DYNAMICS.

1848.

1. A GIVEN inelastic mass is let fall from a given height on one scale of a balance, and two inelastic masses are let fall from different heights on the other scale, so that the three impacts take place simultaneously: find the relations between the masses and heights in order that the balance may remain permanently at rest.

Let M be the given mass, h the height from which it falls; M_1 , M_2 the other two masses, h_1 , h_2 the heights from which they fall: then the momenta of the three will be

$$M(2gh)^{\frac{1}{2}}, M_1(2gh_1)^{\frac{1}{2}}, M_2(2gh_2)^{\frac{1}{2}} \text{ respectively:}$$

in order that equilibrium may not be disturbed, we must have

$$\text{sum of momenta of } M_1, M_2 = \text{momentum of } M,$$

$$\text{or } M_1 h_1^{\frac{1}{2}} + M_2 h_2^{\frac{1}{2}} = M h^{\frac{1}{2}} \dots\dots\dots (1).$$

Also, in order that the balance may remain permanently at rest, we must have

$$M_1 + M_2 = M \dots\dots\dots (2):$$

(1) and (2) are the required relations.

2. A cannon-ball is fired at a mark at a place whose north latitude is l ; shew that in consequence of the Earth's rotation the vertical plane containing the axis of the cannon must be inclined at an angle of $15t \sin l$ seconds to the left of the vertical plane passing through the mark, t being the time of flight in seconds.

The Earth's motion (ω) of rotation about its axis of figure may be resolved into two; one about the vertical line at the

place in question, and another about an axis through the Earth's centre at right angles to the former. The latter rotation will not affect the relative position of the cannon and mark.

The former velocity of rotation is $\omega \sin l$, and carries the mark round the cannon from right to left: consequently the vertical plane through the cannon and mark will in time t revolve through an angle $\omega \sin l.t$, or $\frac{360}{12.60.60} \sin l.t$ degrees, if t be the number of seconds in the time of flight, or through an angle $\frac{360}{12} \sin l.t$, or $15 \sin l.t$ seconds: in order, therefore, that the ball may hit the mark, it must be aimed $15 \sin l.t$ seconds to the left of the mark.

3. An imperfectly elastic homogeneous rough sphere is projected obliquely, without rotation, against a fixed plane; if i, i' be the angles of incidence and reflexion, λ the coefficient of elasticity for direct impact, and ρ the ratio of the tangential force of restitution and compression, prove that

$$2\rho = 5 - 7\lambda \tan i' \cot i.$$

Let R, R_1 be the normal impulses up to the time of greatest compression and during the whole impact respectively:

F, F_1 the same tangential impulses,

V, V' the velocities of the centre of the sphere before and after impact;

$$\therefore R = V \cos i.M,$$

$$R_1 = (1 + \lambda) R,$$

$$\text{and } V' \cos i' = \frac{R_1}{M} - V \cos i = \lambda V \cos i \dots\dots (1).$$

Also tangential velocity before impact $= V \sin i$; and at the time of greatest tangential compression the tangential action F has diminished the velocity $V \sin i$ by the quantity $\frac{F}{M}$, and has generated an angular velocity ω where

$$Mk^2\omega = Fa.$$

We must also express the geometrical condition that the point in contact with the plane is at rest, or

$$V \sin i - \frac{F}{M} = a\omega.$$

$$\text{Hence } V \sin i - \frac{F}{M} = \frac{Fa^2}{Mk^2};$$

$$\therefore F = \frac{k^2}{a^2 + k^2} MV \sin i.$$

$$\text{Also } F_1 = (1 + \rho) F;$$

$$\therefore V' \sin i' = V \sin i - \frac{F_1}{M}$$

$$= V \sin i \left\{ 1 - (1 + \rho) \frac{k^2}{a^2 + k^2} \right\}$$

$$= V \sin i \frac{a^2 - \rho k^2}{a^2 + k^2} \dots \dots \dots (2):$$

(2) \div (1) gives

$$\tan i' = \frac{\tan i}{\lambda} \frac{a^2 - \rho k^2}{a^2 + k^2};$$

or, substituting $\frac{2}{5}a^2$ for k^2 ,

$$\tan i' = \frac{\tan i}{\lambda} \cdot \frac{5 - 2\rho}{7};$$

$$\therefore 2\rho = 5 - 7\lambda \tan i' \cot i.$$

4. Two given masses are connected by a slightly elastic string, and projected so as to whirl round: find the time of a small oscillation in the length of the string. Give a numerical result, supposing the masses to weigh 1 lb., 2 lbs. respectively, and the natural length of the string to be 1 yard, and supposing that it stretches $\frac{1}{10}$ inch for a tension of 1 lb.

The tension, and therefore the extension, of the string will evidently depend only upon the relative motion of the masses, not upon their absolute motions. Now the relative motion of the masses will not be affected if we apply at each instant to

both bodies (M and M') the accelerating force $\frac{T}{M}$, equal to that acting upon M , and in the opposite direction: and if, further, we apply at the instant after projection to both bodies the same velocity, viz. a velocity equal to M 's velocity of projection in the opposite direction to it. But by these means M is reduced to rest: let it be taken as the pole of coordinates. Then, if

$\frac{1}{u_0}$ = the unextended length of the string,

$\frac{1}{u}$ = the extended length of the string at time t ,

and T = tension at time t ,

we have, by the principle that 'Tension \propto Extension',

$$T = E \frac{\frac{1}{u} - \frac{1}{u_0}}{\frac{1}{u_0}} = E \frac{u_0 - u}{u}, \quad E \text{ a constant weight.}$$

Now the equation of M 's motion is

$$\frac{d^2 u}{d\theta^2} + u - \frac{\frac{T}{M} + \frac{T}{M'}}{h^2 u^2} = 0,$$

$$\text{or } \frac{d^2 u}{d\theta^2} + u - E \left(\frac{1}{M} + \frac{1}{M'} \right) \frac{u_0 - u}{h^2 u^3} = 0.$$

Let $u = u_0 - \alpha$, α will be very small,

$$\text{and } \frac{d^2 \alpha}{d\theta^2} - u_0 + \alpha + E \left(\frac{1}{M} + \frac{1}{M'} \right) \frac{\alpha}{h^2 u_0^3} \left(1 + 3 \frac{\alpha}{u_0} \right) = 0;$$

or, omitting α^2 ,

$$\frac{d^2 \alpha}{d\theta^2} + \left\{ 1 + \frac{E}{h^2 u_0^3} \left(\frac{1}{M} + \frac{1}{M'} \right) \right\} \alpha - u_0 = 0;$$

which equation shews that α undergoes periodic inequalities, whose period

$$T = 2\pi \frac{1}{\left\{ 1 + \frac{E}{h^2 u_0^3} \left(\frac{1}{M} + \frac{1}{M'} \right) \right\}^{\frac{1}{2}}};$$

this is the time of the small oscillations in the length of the string: h must be determined by the circumstances of projection.

Ex. Let $M = 1$ lb., $M' = 2$ lb., $\frac{1^2}{u_0} = 1$ yard, and E such a weight that a tension of 1 lb. stretches the string $\frac{1}{10}$ inch, or

$$1 \text{ lb.} = E \frac{1 \text{ yd.} - \frac{1}{10} \text{ in.}}{1 \text{ yd.}} = E \frac{359}{360},$$

$$\text{or } E = \frac{360}{359} \text{ lbs.};$$

$$\begin{aligned} \therefore T &= 2\pi \frac{1}{\left(1 + \frac{1}{h_2 u_0^3} \cdot \frac{360}{359} \cdot \frac{3g}{2}\right)^{\frac{1}{2}}} \text{ seconds} \\ &= 2\pi \frac{1}{\left(1 + \frac{32,2 \times 3}{v^2} \cdot \frac{360}{359} \cdot \frac{3}{2}\right)^{\frac{1}{2}}} \text{ seconds,} \end{aligned}$$

where v is the velocity of projection, expressed in yards, of M or M' in their relative orbits,

$$= \frac{6}{\left(1 + \frac{145}{v^2}\right)^{\frac{1}{2}}} \text{ seconds very nearly.}$$

5. A rough sphere rolls within a hollow cylinder with its axis vertical, so as to be in contact with the curved surface and the flat bottom: find the reactions and the frictions, in terms of the angular velocity with which the sphere goes round, and explain the indeterminateness of the problem.

Let ω be the angular velocity of the centre of the sphere about the axis of the cylinder;

$\omega_x, \omega_y, \omega_z$ the angular velocities of the sphere about that axis and two other axes of rectangular coordinates;

R_x, R_y, R_z and R'_x, R'_y, R'_z the mutual actions at the base and other points of contact parallel to the axes of x, y , and z respectively;

a and $r + a$ the radii of the sphere and cylinder.

Then the equations of motion are

$$M \frac{d^2 x}{dt^2} = R_x + R'_x,$$

$$M \frac{d^2 y}{dt^2} = R_y + R'_y,$$

$$M \frac{d^2 z}{dt^2} = R_z + R'_z - Mg = 0,$$

$$Mk^2 \frac{d\omega_x}{dt} = -R_y a - R'_z a \sin \theta,$$

(θ the angle between the radius vector and axis of x .)

$$Mk^2 \frac{d\omega_y}{dt} = R_x a + R'_z a \cos \theta,$$

$$Mk^2 \frac{d\omega_z}{dt} = R'_x a \sin \theta - R'_y a \cos \theta.$$

The geometrical condition to be expressed is, that the two points of contact must be instantaneously at rest.

Hence,

horizontal motion of the point of contact of the curve surfaces, or

$$a\omega_x - r\omega = 0;$$

vertical motion of the same, or

$$a\omega_x \sin \theta - a\omega_y \cos \theta = 0;$$

and horizontal motion of the other point, or

$$a\omega_x \cos \theta + a\omega_y \sin \theta - r\omega = 0.$$

Hence,

$$a\omega_x = r\omega \cos \theta,$$

$$a\omega_y = r\omega \sin \theta;$$

$$\therefore a \frac{d\omega_x}{dt} = r \cos \theta \cdot f - r \sin \theta \cdot \omega^2,$$

(where $f = \frac{d\omega}{dt}$),

$$a \frac{d\omega_y}{dt} = r \sin \theta . f + r \cos \theta . \omega^2,$$

$$a \frac{d\omega_z}{dt} = r f.$$

Also,

$$x = r \cos \theta,$$

$$y = r \sin \theta ;$$

$$\therefore \frac{d^2 x}{dt^2} = - r \sin \theta . f - r \cos \theta . \omega^2,$$

$$\frac{d^2 y}{dt^2} = r \cos \theta . f - r \sin \theta . \omega^2.$$

Hence the equations of motion become

$$- Mr \sin \theta . f - Mr \cos \theta . \omega^2 = R_x + R'_x \dots \dots \dots (1),$$

$$Mr \cos \theta . f - Mr \sin \theta . \omega^2 = R_y + R'_y \dots \dots \dots (2),$$

$$0 = R_z + R'_z - Mg \dots \dots \dots (3),$$

$$Mk^2 r \cos \theta . f - Mk^2 r \sin \theta . \omega^2 = - R_y a^2 - R'_x a^2 \sin \theta \dots \dots (4),$$

$$Mk^2 r \sin \theta . f + Mk^2 r \cos \theta . \omega^2 = R_x a^2 + R'_z a^2 \cos \theta \dots \dots (5),$$

$$Mk^2 r f = R'_x a^2 \sin \theta - R'_y a^2 \cos \theta \dots (6),$$

$$(4) \cos \theta + (5) \sin \theta + (6),$$

$$2Mk^2 r f = (R_x + R'_x) a^2 \sin \theta - (R_y + R'_y) a^2 \cos \theta$$

$$= - Ma^2 r f, \text{ by (1) and (2);}$$

$$\therefore f = 0,$$

and ω is constant.

Hence (4) $\cos \theta + (5) \sin \theta$,

$$0 = R_x \sin \theta - R_y \cos \theta,$$

and (1) $\sin \theta - (2) \cos \theta$,

$$0 = R'_x \sin \theta - R'_y \cos \theta.$$

Hence there is no action perpendicular to the plane through the radius vector and the axis of the cylinder. Let R and R' be the horizontal pressures in that plane on the base and at

the other point of contact. Then (1) $\cos \theta + (2) \sin \theta$ and (5) $\cos \theta - (4) \sin \theta$ give

$$R + R' = -Mr\omega^2 \dots\dots\dots (7),$$

$$\text{and } R + R'_z = M \frac{h^2}{a^2} r\omega^2 \dots\dots\dots (8):$$

(3), (7), and (8) are the only equations for determining R, R', R_z , and R'_z .

The indeterminateness of the problem arises from the circumstance, that there are more pressures on the sphere than are necessary to produce the motion required. Thus, there are the two vertical forces R_z and R'_z to support the weight, the two radial forces R and R' to curve the path of the centre of the sphere, and the two, R and R'_z , to oppose the tendency of the sphere to rotate about a horizontal axis perpendicular to the radius-vector. Hence the above equations contain the sums of couples of these quantities. Considerations of elasticity, which prevents all such ambiguities in nature, would remove them from the solution of the problem.

6. A uniform bent lever, whose arms are at right angles to each other, is capable of being enclosed in the interior of a smooth spherical surface; determine the position of equilibrium.

Find also the time of a small oscillation when the position of equilibrium is slightly disturbed.

Since the reactions of the sphere all pass through the centre, it is plain that the resultant force of gravity upon the lever must also pass through the centre of the sphere; hence, its centre of gravity must lie vertically under the centre of the sphere.

Let C (fig. 101) be the angle of the lever ACB , join AB ; bisect AB, AC, BC , in O, D , and E , and in ED take the point G , such that $EG:ED::AC:AC+BC$: join OE, OG, OD : G will be the centre of gravity of the lever, and OD, OE , will be perpendicular to AC, BC : also O will be

the centre of the sphere. Hence we must have OG vertical. But $EG : GD :: CD : EC :: OE : OD$; therefore OG bisects the right angle OD , and AC, BC , are equally inclined to the horizon.

When the lever is slightly disturbed in its own (vertical) plane from its position of equilibrium, it will manifestly oscillate as if it were attached to an axis through O , and the sphere removed. Hence we have to find the radius (k) of gyration of the straight lines AC, BC , about an axis through O , perpendicular to the plane of BC, CA .

Let k_1, k_2 , be the radii for AC, BC , respectively: then

$$\begin{aligned} k_1^2 &= OD^2 + \frac{1}{3}AD^2 = EC^2 + \frac{1}{3}AD^2 \\ &= b^2 + \frac{1}{3}a^2, \quad \text{if } AC = 2a, \quad BC = 2b. \end{aligned}$$

Similarly,

$$k_2^2 = a^2 + \frac{1}{3}b^2,$$

$$\text{and } k^2 = \frac{ak_1^2 + bk_2^2}{a+b} = \frac{\frac{1}{3}a^3 + ab^2 + a^2b + \frac{1}{3}b^3}{a+b} = \frac{1}{3}(a+b)^2.$$

Again, to find $OG (=l)$, the distance of the centre of gravity of the lever from O . We have

$$\begin{aligned} l &= EG \cdot \frac{\sin OEG}{\sin EOG} = \frac{a}{a+b} ED \frac{\sin OED}{\sin EOG} \\ &= \frac{2^{\frac{1}{2}}ab}{a+b}; \end{aligned}$$

therefore the time of a small oscillation

$$\begin{aligned} &= 2\pi \frac{k}{(gl)^{\frac{1}{2}}} \\ &= \frac{2^{\frac{3}{2}}}{3^{\frac{1}{2}}} \pi \frac{(a+b)^{\frac{3}{2}}}{(gab)^{\frac{1}{2}}}. \end{aligned}$$

7. A section of the surface of a circular right cone (whose axis is horizontal and vertical angle 60°) is formed by a plane perpendicular to the slant side, so as to contain the vertex;

shew that when the surface so cut off makes small oscillations about the axis, the length of the isochronous pendulum $= \frac{21a}{16}$ (whether the elliptic base be included in the surface or not), a being the length of the perpendicular drawn from the vertex upon the elliptic base.

Let k be the radius of gyration of the section about the axis;

$$\therefore k^2 = \frac{\Sigma \delta S r^2}{\Sigma \delta S},$$

r being the distance of the element δS of the surface from the axis. Let δS be projected upon a plane perpendicular to the axis, and $\delta S'$ be the corresponding elementary surface;

$$\therefore \delta S' = \delta S \cos 30^\circ;$$

$$\therefore k^2 = \frac{\Sigma \delta S' r^2}{\Sigma \delta S'}$$

= the square of the radius of gyration of the elliptic projection on a plane perpendicular to the axis.

Similarly, the radius of gyration of the elliptic base equals the radius of gyration of its projection on the same plane, which is the same as the projection of the whole section. To find this radius we must first find the axes of the elliptic base BC (fig. 102).

Since the vertical angle $BAC = 60^\circ$, we have, if $AC = a$,

$$BC = \frac{3^{\frac{1}{2}}}{2} a;$$

and if b equals the semi-axis minor,

$$(2b)^2 = CF.BE = a.2a;$$

$$\therefore b^2 = \frac{1}{2}a^2.$$

We may now also find Oo , the distance of o , the centre of the base from O , the point where the axis pierces the base. We have

$$\begin{aligned} Oo &= oC - OC \\ &= \frac{3^{\frac{1}{2}}}{2} a - \frac{1}{3^{\frac{1}{2}}} a; \end{aligned}$$

$$\begin{aligned}\therefore Oo^2 &= \frac{1}{12}a^2, \\ \text{and } Oo'^2 &= \frac{3}{4}Oo^2 = \frac{1}{16}a^2, \\ \text{or } Oo' &= \frac{1}{4}a.\end{aligned}$$

Let a' , b' , be the semi-axes of the ellipse BD , which is the projection upon the plane BE of the conical surface, as well as of the elliptic base BC ;

$$\begin{aligned}\therefore 2a' &= BD = CF + \frac{1}{2}(BE - CF) \\ &= a + \frac{1}{2}a = \frac{3}{2}a,\end{aligned}$$

$$\text{and } a' = \frac{3}{4}a,$$

$$\begin{aligned}\text{and } b' &= \text{semi-axis minor of } BC \\ &= \frac{1}{2^{\frac{1}{2}}}a.\end{aligned}$$

Hence the radius² of gyration of BD about Oo'

$$\begin{aligned}&= \frac{1}{4}(a'^2 + b'^2) \\ &= \frac{1}{4}\left(\frac{9}{16} + \frac{1}{2}\right)a^2 \\ &= \frac{17}{4.16}a^2;\end{aligned}$$

and the length of the simple pendulum of BD about the axis

$$\begin{aligned}&= \frac{17}{4.16} \frac{a^2}{Oo'} + Oo' \\ &= \frac{17}{16}a + \frac{1}{4}a \\ &= \frac{21}{16}a.\end{aligned}$$

Since the length of the simple pendulum for the conical surface is the same as for the elliptic base, it is the same for the conical surface alone and taken with the base.

1849.

1. If a uniform inextensible string, in the form of any continuous curve, be subjected to an impulsive tension at its extremities, the tension at any point will vary directly as the velocity communicated to that point in the direction of the radius of absolute curvature, and inversely as the curvature.

Let T be the tension at any point, then $T \frac{dx}{ds}$, $T \frac{dy}{ds}$, $T \frac{dz}{ds}$, are the tensions in directions of the axes; and since the tensions are impulsive, we have

difference of tensions at the extremities of any small arc
 \propto velocity communicated to the arc;

$$\therefore d\left(T \frac{dx}{ds}\right) \propto \frac{dx}{dt},$$

$$\text{or } \frac{dx}{ds} dT + T \frac{d^2x}{ds^2} ds \propto \frac{dx}{dt} \dots\dots\dots (1).$$

$$\text{Similarly, } \frac{dy}{ds} dT + T \frac{d^2y}{ds^2} ds \propto \frac{dy}{dt} \dots\dots\dots (2),$$

$$\frac{dz}{ds} dT + T \frac{d^2z}{ds^2} ds \propto \frac{dz}{dt} \dots\dots\dots (3).$$

(1) $\frac{d^2x}{ds^2}$ + (2) $\frac{d^2y}{ds^2}$ + (3) $\frac{d^2z}{ds^2}$ gives

$$T \propto \frac{\frac{dx}{dt} \frac{d^2x}{ds^2} + \frac{dy}{dt} \frac{d^2y}{ds^2} + \frac{dz}{dt} \frac{d^2z}{ds^2}}{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2},$$

$$\text{since } \frac{d^2x}{ds^2} \frac{dx}{ds} + \frac{d^2y}{ds^2} \frac{dy}{ds} + \frac{d^2z}{ds^2} \frac{dz}{ds} = 0.$$

Now, the direction-cosines of the radius of curvature are

$$\frac{\frac{dx}{ds}}{\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}^{\frac{1}{2}}}, \quad \frac{\frac{dy}{ds}}{\dots}, \quad \frac{\frac{dz}{ds}}{\dots}.$$

Also, if ρ be the radius of curvature,

$$\frac{1}{\rho} = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}^{\frac{1}{2}};$$

$\therefore T \propto$ velocity communicated to (xyz) in the direction of the radius of absolute curvature, and inversely as the curvature.

2. The nut of a screw rests upon a smooth horizontal plane, over a hole cut so as to allow a free passage for the screw, and the screw descends through the nut by its own weight: determine the motion.

At time t let P be the whole action between the screw and nut perpendicular to the thread of the screw, which makes an angle α suppose with the horizon. Then

the whole vertical force on the screw $= Mg - P \cos \alpha$,
 moment of the whole horizontal force $= Pa \sin \alpha$,
 on the nut $= - Pa \sin \alpha$.

Hence, if y = depth of any point of the screw below a fixed plane,

ω, ω' , the angular velocities of the screw and nut,

$$\frac{d^2 y}{dt^2} = g - \frac{P \cos \alpha}{M},$$

$$k^2 \frac{d\omega}{dt} = \frac{Pa \sin \alpha}{M},$$

$$k'^2 \frac{d\omega'}{dt} = - \frac{Pa \sin \alpha}{M'}$$

The geometrical condition is, that each two corresponding points of the screw and nut in contact have the same motion perpendicular to the thread;

$$\therefore a\omega \sin \alpha - \frac{dy}{dt} \cos \alpha = a'\omega' \sin \alpha.$$

Differentiating this equation and substituting from the above,

$$a \sin \alpha \frac{d\omega}{dt} - \cos \alpha \frac{d^2 y}{dt^2} = a' \sin \alpha \frac{d\omega'}{dt};$$

$$\therefore \frac{Pa^2 \sin^2 \alpha}{Mk^2} + \frac{P \cos^2 \alpha}{M} - g \cos \alpha + \frac{Pa'^2 \sin^2 \alpha}{M'k'^2} = 0,$$

whence P is constant, and its value known: by substitution of this value we determine the three required parts of the motion, which thus appear to be uniformly accelerated.

3. The centre of a rough sphere is fixed; if another sphere be placed on the top of it and just displaced, determine the motion of both spheres.

Let O, o , (fig. 103) be the centres of the spheres at the time t ; OC, oc , the two radii which in the beginning of motion were

vertical, so that C, c , coincided: then, calling the different parts and angles, as in the figure, we have for the equations of motion, for the lower sphere,

$$Mk^2 \frac{d^2\theta}{dt^2} = Fa \dots\dots\dots (1).$$

for the upper one,

$$M' \frac{d^2x}{dt^2} = R \sin\phi - F \cos\phi \dots\dots\dots (2),$$

$$M' \frac{d^2y}{dt^2} = R \cos\phi + F \sin\phi - M'g \dots\dots\dots (3),$$

$$M'k'^2 \frac{d^2\theta'}{dt^2} = Fb \dots\dots\dots (4);$$

and for the geometrical condition we must express the circumstance that the spheres roll without sliding;

$$\therefore a(\phi - \theta) = b(\theta' - \phi) \dots\dots\dots (5).$$

Also we have

$$x = (a + b) \sin\phi,$$

$$y = (a + b) \cos\phi.$$

Taking (1) $b - (4) a$, we have

$$Mk^2b \frac{d^2\theta}{dt^2} - M'k'^2a \frac{d^2\theta'}{dt^2} = 0;$$

whence we find

$$\theta' = \frac{Mk^2b}{M'k'^2a} = n\theta \text{ suppose.}$$

Hence (5) becomes

$$a(\phi - \theta) = nb\theta - b\phi,$$

$$\text{or } \theta = \frac{a + b}{a + nb} \phi;$$

$$\text{and } \theta' = n \frac{a + b}{a + nb} \phi.$$

Now the expression for the *vis viva* gives us

$$Mk^2 \left(\frac{d\theta}{dt} \right)^2 + M' \left\{ (a + b)^2 \left(\frac{d\phi}{dt} \right)^2 + k'^2 \left(\frac{d\theta'}{dt} \right)^2 \right\} = 2M'g(a + b - y);$$

and substituting the above values of θ and θ' , we get

$$\begin{aligned} \left[Mk^2 \left(\frac{a+b}{a+nb} \right)^2 + M \left\{ (a+b)^2 + \left(\frac{a+b}{a+nb} \right)^2 n^2 k'^2 \right\} \right] \left(\frac{d\phi}{dt} \right)^2 \\ = 2M'g(a+b-y) \\ = 2M'g(a+b)(1-\cos\phi): \end{aligned}$$

which may be written shortly

$$\left(\frac{d\phi}{dt} \right)^2 = \frac{1}{2} m^2 (1 - \cos\phi) = m^2 \sin^2 \frac{1}{2} \phi;$$

$$\therefore m \frac{dt}{d\phi} = \frac{1}{\sin \frac{1}{2} \phi},$$

$$\text{and } mt + C = 2 \log \tan \frac{1}{4} \phi.$$

The constant C may be determined by supposing ϕ to have a very small value α , when $t = 0$; whence

$$mt = 2 \log \frac{\tan \frac{1}{4} \phi}{\tan \frac{1}{4} \alpha},$$

$$\text{and } \tan \frac{1}{4} \phi = \tan \frac{1}{4} \alpha e^{\frac{1}{2} mt},$$

which determines ϕ , and thence θ and θ' , in terms of t .

4. What must be the angular velocity of a horizontal cylinder, in order that a heavy string of given length attached to it may be just wound up?

Let l be the whole length of the string, x that of the part hanging down, μ the mass of a unit of length; T the tension of the rope at the point of its contact with the cylinder. Then for the equation of motion of the cylinder, and that part of the rope coiled on it,

$$\{Mk^2 + \mu(l-x)a^2\} \frac{d^2\theta}{dt^2} = -Ta \dots\dots\dots (1);$$

and considering the part of the string hanging down as one mass, the coordinate of a fixed point of which (the extremity) is x ,

$$\mu x \frac{d^2x}{dt^2} = \mu xg - T \dots\dots\dots (2),$$

$$\text{and } l - x = a\theta.$$

Now (1) - (2) a gives

$$\{Mk^2 + \mu(l-x)a^2\} \frac{d^2\theta}{dt^2} - \mu ax \frac{d^2x}{dt^2} = -\mu gax,$$

$$\text{or } \{Mk^2 + \mu(l-x)a^2\} \frac{d^2x}{dt^2} + \mu a^2 x \frac{d^2x}{dt^2} = \mu g a^2 x,$$

$$\text{or } (Mk^2 + \mu l a^2) \frac{d^2x}{dt^2} = \mu g a^2 x.$$

Multiply by $2 \frac{dx}{dt}$, and integrate,

$$\therefore (Mk^2 + \mu l a^2) \left(\frac{dx}{dt}\right)^2 = \mu g a^2 x^2 + C:$$

and $\frac{dx}{dt} = 0$, when $x = 0$; $\therefore C = 0$;

$$\therefore (Mk^2 + \mu l a^2) \left(\frac{dx}{dt}\right)^2 = \mu g a^2 x^2.$$

Hence, in the beginning of motion, when $x = l$,

$$\frac{d\theta}{dt} = -\frac{1}{a} \frac{dx}{dt} = \frac{(\mu g)^{\frac{1}{2}} l}{(Mk^2 + \mu l a^2)^{\frac{1}{2}}},$$

which is the required angular velocity.

5. A heavy rod is suspended from a fixed point by two inextensible strings without weight, the strings and the rod forming an equilateral triangle; if either of the strings be cut, determine the initial tension of the other.

Let the figure (104) represent the position of the beam at the time t after the string has been cut; CN being the vertical line through the point of support. Hence the equations of motion will be

$$M \frac{d^2x}{dt^2} = T \sin \theta \dots\dots\dots (1),$$

$$M \frac{d^2y}{dt^2} = Mg - T \cos \theta \dots\dots\dots (2),$$

$$Mk^2 \frac{d^2\phi}{dt^2} = -Ta \sin(\theta + \phi) \dots\dots\dots (3).$$

Also the geometry gives us

$$x = 2a \sin \theta - a \sin \phi,$$

$$y = 2a \cos \theta + a \cos \phi.$$

Hence, differentiating twice, we find

$$\begin{aligned}\frac{d^2x}{dt^2} \sin\theta + \frac{d^2y}{dt^2} \cos\theta &= -2a \left(\frac{d\theta}{dt}\right)^2 - a \sin(\theta + \phi) \frac{d^2\phi}{dt^2} \\ &\quad - a \cos(\theta + \phi) \left(\frac{d\phi}{dt}\right)^2 \\ &= g \cos\theta - \frac{T}{M} \cos 2\theta,\end{aligned}$$

by (1) and (2).

Hence we find, by substituting the value of $\frac{d^2\phi}{dt^2}$ from (3),

$$\frac{T}{M} \{\cos 2\theta + \frac{a^2}{k^2} \sin^2(\theta + \phi)\} = g \cos\theta + 2a \left(\frac{d\theta}{dt}\right)^2 + a \cos(\theta + \phi) \left(\frac{d\phi}{dt}\right)^2.$$

Now, in the beginning of motion, $\frac{d\theta}{dt} = 0$, $\frac{d\phi}{dt} = 0$, $\theta = 30^\circ$, and $\phi = 90^\circ$: let T_0 be the initial tension;

$$\therefore T_0 \left(\frac{1}{2} + \frac{a^2}{k^2} \cdot \frac{3}{4}\right) = Mg \frac{3^{\frac{1}{2}}}{2};$$

and $k^2 = \frac{1}{3}a^2$;

$$\therefore T_0 \left(\frac{1}{2} + \frac{9}{4}\right) = Mg \frac{3^{\frac{1}{2}}}{2},$$

$$\text{or } T_0 = \frac{2 \cdot 3^{\frac{1}{2}}}{11} Mg,$$

the required tension.

6. A man standing in a swing is set in motion: shew that he can accelerate the motion and increase the arc of oscillation by crouching and rising in the swing; and prove that the effect will be greatest if he crouch when the swing is at the highest point, and rise when it is at the lowest point of its arc of oscillation.

Since the ropes of the swing are not supposed to slacken or bend, we may suppose them to become rigid, and rigidly connected with the swing.

If the man do not crouch and rise, the arc of oscillation will be unaltered, the effect of gravity being to accelerate the motion while he is descending, and to retard it while he ascends.

Now, if the man rises when the swing is at its lowest point, the moment of the force of gravity on him about the axis through the points of support of the swing is diminished, and the motion less retarded than it would have been if he had retained a mean position; hence the swing will rise higher than it otherwise would: if he crouches when at the highest point of the arc of oscillation, the motion will be more accelerated while the swing descends than it would have been if he had remained in a mean position; hence the velocity at the lowest point will be increased on account of his having both crouched and risen; hence the arc of oscillation will be increased by such a motion of his body.

It is evident that it will be most increased if he rises at the lowest and crouches at the highest point of the arc of oscillation.

In addition to the above reasons why the supposed motion of crouching and rising will increase the arc of oscillation, is another, viz. that the principle of the conservation of areas must hold during the sudden motion of rising at the lowest point. For during that motion both the forces on the man, viz. gravity and the upward pressure of the swing, may be considered as acting in a vertical direction, that is, normally to his instantaneous direction of motion. The consequence will be, that his linear velocity will be increased as he rises, and therefore approaches the horizontal axis through the points of support of the ropes.

If the swing be supposed to have mass this effect will be diminished, since his rising will not so much raise the common centre of gravity of himself and the swing. This diminution of the effects of the principle of conservation of areas will be practically caused by a change of the friction between the swing and his feet, which will for the instant retard his motion more than it usually does.

The above reasoning has, of course, no place as applied to his crouching when at the highest point of the arc of oscillation, since he is then describing no areas at all about the horizontal axis through the points of support.

7. A circular hoop rests upon a smooth horizontal plane with a particle at its lowest point, and receives a horizontal velocity of projection V in its own plane: find the value of V in order that the particle may just rise to the height of the centre of the hoop.

Determine the motion when V is greater, and also when it is less than this value; and find the time of an oscillation of the particle in the hoop when V is small.

Let P be the position of the particle in the hoop (fig. 105) at the time t from beginning of motion. Let $AN = x$, $NP = y$, be the coordinates of P referred to A , its position at projection, as origin, $AM = x'$. The principle of the conservation of the motion of the centre of gravity in a horizontal direction gives us

$$M \frac{dx}{dt} + M' \frac{dx'}{dt} = M' V \dots\dots\dots (1).$$

The expression for *vis viva* is

$$M \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\} + M' \left(\frac{dx'}{dt} \right)^2 = M' V^2 - 2Mgy \dots (2).$$

Also

$$x' - x = a \sin \theta,$$

$$y = a (1 - \cos \theta);$$

$$\therefore \frac{dx'}{dt} - \frac{dx}{dt} = a \cos \theta \frac{d\theta}{dt} \dots\dots\dots (3),$$

$$\frac{dy}{dt} = a \sin \theta \frac{d\theta}{dt} \dots\dots\dots (4).$$

From (1) and (3),

$$(M + M') \frac{dx}{dt} = M' \left(V - a \cos \theta \frac{d\theta}{dt} \right),$$

$$(M + M') \frac{dx'}{dt} = M' V + Ma \cos \theta \frac{d\theta}{dt}.$$

Hence equation (2) becomes

$$\frac{M'}{(M + M')^2} \left\{ MM' \left(V - a \cos \theta \frac{d\theta}{dt} \right)^2 + \left(M' V + Ma \cos \theta \frac{d\theta}{dt} \right)^2 \right\} \\ = M' V^2 - 2Mgy,$$

$$\text{or } \frac{M'}{M + M'} \left\{ M' V^2 + Ma^2 \cos^2 \theta \left(\frac{d\theta}{dt} \right)^2 \right\} = M V^2 - 2Mgy \dots (5).$$

If the particle just rises to the height of the centre, $\frac{d\theta}{dt} = 0$, when $\theta = 90^\circ$ and $y = a$;

$$\therefore \left(M' - \frac{M'^2}{M + M'} \right) V^2 = 2Mga,$$

$$\text{or } V = \left(\frac{M + M'}{M'} \right)^{\frac{1}{2}} (2ga)^{\frac{1}{2}}.$$

When V is greater than this value, equation (5) gives the height to which it will rise before $\frac{d\theta}{dt} = 0$, viz.

$$y = \frac{M'}{M + M'} \cdot \frac{V^2}{2g}.$$

At this time the particle is moving horizontally with the same velocity as the hoop: it will now fall down in a parabolic path and will strike the hoop at a point at the same distance below the horizontal diameter as the point at which it left the hoop is above it.

If V is not sufficiently great to make it rise to the height of the centre, it will rise to the height

$$y = \frac{M'}{M + M'} \cdot \frac{V^2}{2g};$$

and since the above equations apply for both directions of motion of the particle, we see that it will continue to oscillate, rising on both sides of the vertical diameter to the above height.

If V be very small, θ and $\frac{d\theta}{dt}$ will be very small: in this case differentiate (5);

$$\begin{aligned} \therefore \frac{M'a^2}{M + M'} \left\{ \cos^2 \theta \cdot 2 \frac{d^2 \theta}{dt} \frac{d\theta}{dt} - 2 \sin \theta \cos \theta \left(\frac{d\theta}{dt} \right)^3 \right\} &= -2g \frac{dy}{dt} \\ &= -2ga \sin \theta \frac{d\theta}{dt}, \end{aligned}$$

$$\text{or } \frac{M'a}{M + M'} \left\{ (1 - \theta^2) \frac{d^2 \theta}{dt^2} - \theta \left(\frac{d\theta}{dt} \right)^2 \right\} = -g\theta;$$

or omitting θ^2 and $\theta \left(\frac{d\theta}{dt} \right)^2$,

$$\frac{d^2\theta}{dt^2} + \frac{M+M'}{M'} \frac{g}{a} \cdot \theta = 0,$$

the equation of oscillatory motion, the time of oscillation being

$$2\pi \left(\frac{M'}{M+M'} \cdot \frac{a}{g} \right)^{\frac{1}{2}}.$$

8. A heavy lamina, in the form of an equilateral triangle, suspended from a fixed point by three equal strings, is drawn a little aside from its horizontal position (the strings being all stretched); its centre of gravity then receives a small horizontal velocity of projection perpendicular to the plane in which the displacement was made, while at the same time a velocity of rotation is communicated to it in its own plane: determine the motion.

The motion of rotation of the lamina in its own plane is evidently independent of the motion of its centre of gravity, and will continue uniform.

By the principle of the superposition of small motions, the oscillations of the centre of gravity in the two perpendicular planes will be independent of each other.

The equations of these small motions will be

$$\theta = \theta_1 \cos nt$$

$$\text{and } \phi = \phi_1 \sin nt,$$

θ_1 and ϕ_1 being the semi-arcs of oscillation;

$$\therefore \frac{\theta^2}{\theta_1^2} + \frac{\phi^2}{\phi_1^2} = 1,$$

and the centre of gravity will move very nearly in a small ellipse about its position of equilibrium as centre, with axes $r\theta_1$ and $r\phi_1$, r being the distance of the centre of gravity from the point of suspension.

9. A man hangs by a rod which swings in a vertical plane: compare the exertion required to raise him from one given point of the rod to another, 1st, when he draws himself through a given small space always when the rod is vertical, and 2ndly, if he makes the effort when the rod is at its greatest inclination to the vertical.

See Prob. 6, 1849.

Let v be the velocity of the man at the lowest point before he begins to raise himself, 2α the arc of oscillation: then, if a be his original distance from the point of support,

$$v^2 = a(1 - \cos \alpha).$$

Now, if he raises himself always through the given small space δ at the lowest point of the arc, the velocity at the lowest point will receive a sudden increase; and it will be with this increased velocity that he will next arrive at the lowest point: the arc of oscillation will also continually increase.

Let v_r be the velocity with which the man arrives for the r^{th} time at the lowest point of the arc, when he raises himself through the r^{th} small space δ . The exertion of doing this

$$= M \left\{ g + \frac{v_r^2}{a - (r-1)\delta} \right\} \delta.$$

To determine v_r . The relation between v_r and v_{r-1} is given us by the equation

$$v_r \{a - (r-1)\delta\} = v_{r-1} \{a - (r-2)\delta\},$$

which expresses the conservation of areas during the man's rise through the small space δ . We may hence deduce the equation

$$v_r \{a - (r-1)\delta\} = va.$$

This equation we may also derive from the consideration, that the man returns each successive time to the point when he raises himself with the velocity with which he quitted it; and therefore we may consider that he raises himself by one effort through the space $(r-1)\delta$, the conservation of areas holding all the while. This consideration gives us the above equation immediately. We thus have

$$\frac{v_r^2}{a - (r-1)\delta} = \frac{v^2 a^2}{\{a - (r-1)\delta\}^3}.$$

Hence the r^{th} exertion

$$= M \left\{ g + \frac{v^2 a^2}{\{a - (r-1)\delta\}^3} \right\} \delta.$$

Let us call $(r-1)\delta$, x ; then we may call δ , dx , and we shall have, as an approximation to the true result, supposing δ extremely small,

whole exertion

$$\begin{aligned}
 &= M \left\{ gh + v^2 a^2 \int_0^h \frac{dx}{(a-x)^3} \right\} \\
 &= M \left\{ gh + \frac{1}{2} v^2 a^2 \left(\frac{1}{(a-h)^2} - \frac{1}{a^2} \right) \right\} \\
 &= M \left\{ g + \frac{v^2 (2a-h)}{2(a-h)^2} \right\} h.
 \end{aligned}$$

If the man raises himself at the highest point of the arc, the arc of oscillation will remain unaltered, but the velocity at the lowest point will be increased after each effort. Hence every exertion will be the same, viz. $Mg \sin \alpha \cdot \delta$, and the whole exertion $Mg \sin \alpha \cdot h$.

Hence the ratio of the two exertions

$$= \left(1 + \frac{v^2 (2a-h)}{2g(a-h)^2} \right) \operatorname{cosec} \alpha.$$

10. A semicircular board, moving in its own plane without rotation, and with its curved boundary foremost, comes in contact with a smooth fixed obstacle: determine at what point the impact should take place in order that the angular velocity generated may be the greatest possible.

Let P (fig. 106) be the point where the impact should take place, the radius CP making an angle θ with CD the bisecting radius. Let V be the velocity (in CD) of the centre of gravity of the board before impact, v, v' , those parallel to CD and CB after impact, ω the angular velocity after impact, R the impulse: then, if G be the centre of gravity and $CG = \alpha$,

$$\left. \begin{aligned}
 v &= V - \frac{R}{M} \cos \theta, \\
 v' &= \frac{R}{M} \sin \theta, \\
 \omega &= \frac{R\alpha \sin \theta}{Mk^2},
 \end{aligned} \right\} \dots\dots\dots (\Lambda).$$

Also we have the geometrical condition, that P must have no motion in the direction CP after impact; whence

$$v \cos \theta - v' \sin \theta - \varpi PG \sin GPC = 0,$$

$$\text{or } v \cos \theta - v' \sin \theta - \varpi \alpha \sin \theta = 0.$$

Finding v, v' , from equations (Λ) in terms of ϖ , and substituting in this last equations, we get

$$\left(V - \frac{k^2 \varpi}{\alpha} \cot \theta \right) \cos \theta - \frac{k^2 \varpi}{\alpha} \sin \theta - \varpi \alpha \sin \theta = 0,$$

$$\text{or } V \cos \theta - \left(\frac{k^2}{\alpha} \frac{1}{\sin \theta} + \alpha \sin \theta \right) \varpi = 0;$$

$$\therefore \varpi = \frac{V \sin \theta \cos \theta}{\frac{k^2}{\alpha} + \alpha \sin^2 \theta},$$

which is to be a maximum by the variation of θ .

$$\text{Now, } \alpha = \frac{4}{3\pi} a,$$

$$k^2 = \frac{1}{2} a^2 - \alpha^2;$$

$$\therefore \varpi = \frac{V \alpha \sin \theta \cos \theta}{\frac{1}{2} a^2 - \alpha^2 \cos^2 \theta}$$

$$= \frac{V \alpha \sin 2\theta}{a^2 - \alpha^2 (1 + \cos 2\theta)}.$$

Taking the logarithmic differential,

$$0 = \cot 2\theta - \frac{\alpha^2 \sin 2\theta}{a^2 - \alpha^2 (1 + \cos 2\theta)},$$

$$\text{or } (\alpha^2 - a^2) \cot 2\theta - \alpha^2 \operatorname{cosec} 2\theta = 0,$$

$$\text{or } \cos 2\theta = \frac{\alpha^2}{a^2 - \alpha^2}$$

$$= \frac{16}{9\pi^2 - 16},$$

which determines the position of the point P .

11. A uniform solid cylinder is revolving with a given angular velocity about its centre of gravity, which is fixed; the

cylinder then receives a blow of given intensity in a direction perpendicular to the plane in which the axis moves: determine the subsequent motion.

Since any section of the cylinder through its axis is a principal section, the blow takes place in a principal plane, and therefore only generates a velocity about the axis (that of x suppose) perpendicular to the axis of previous rotation (that of y), and the axis of the cylinder (that of z).

Hence, if A be the moment of inertia about the axes of x and y , C that about the axis of z ; and $\omega_1, \omega_2, \omega_3$, be the angular velocities about these respective axes at any time t after impact, we have as equations of motion,

$$A \frac{d\omega_1}{dt} - (A - C) \omega_2 \omega_3 = 0,$$

$$A \frac{d\omega_2}{dt} - (C - A) \omega_3 \omega_1 = 0,$$

$$C \frac{d\omega_3}{dt} - (A - A) \omega_1 \omega_2 = 0,$$

$$\text{or } \frac{d\omega_3}{dt} = 0,$$

$$\text{and } \omega_3 = \text{constant}$$

$$= 0, \text{ since that is its original value.}$$

$$\text{Hence also } \omega_1 = \text{constant}$$

$$= \text{that generated by the blow}$$

$$= \frac{Ra}{A}, \text{ if } R \text{ be the blow, } a \text{ the distance from}$$

the centre of its point of application,

$$\text{and } \omega_2 = \text{constant}$$

$$= \text{its original value before impact.}$$

Hence the cylinder revolves uniformly, and the instantaneous axis is fixed in it, viz. in the plane xy ; hence this axis is also fixed in space, and the axis of the cylinder, as before, sweeps out a plane.

12. A solid cone is suspended by its vertex from a point in a perfectly rough wall: if the cone be slightly displaced from its equilibrium position, the surface remaining in contact with the wall, determine the time of a small oscillation.

Let θ be the angle which the line of contact of the wall and cone make with the vertical at the time t ; ω the angular velocity at the same time about the line of contact; θ_0 the original value of θ ; h the height and 2α the vertical angle of the cone: then, since the line of contact is the instantaneous axis, the equation of *vis viva* gives us

$$Mk'^2\omega^2 = 2Mg \cdot \frac{3}{4}h \cos\alpha (\cos\theta - \cos\theta_0),$$

$$\text{or} \quad \omega^2 = \frac{3}{2} \frac{gh \cos\alpha}{k'^2} (\cos\theta - \cos\theta_0).$$

Now, to connect ω and θ , we have two expressions for the motion of the centre of the base, viz. $h \sin\alpha \cdot \omega$ and $h \cos\alpha \cdot \frac{d\theta}{dt}$;

$$\therefore \omega = \frac{d\theta}{dt} \cot\alpha;$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = \frac{3}{2} \frac{gh \sin^2\alpha}{k'^2 \cos\alpha} (\cos\theta - \cos\theta_0);$$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{3}{4} \frac{gl \sin^2\alpha}{k'^2} \sin\theta, \quad l \text{ the length of the side};$$

or if the motion be very small,

$$\frac{d^2\theta}{dt^2} + \frac{3}{4} \frac{gl \sin^2\alpha}{k'^2} \theta = 0.$$

Therefore the time of a small oscillation $= 4\pi \frac{k'}{(3gl)^{\frac{1}{2}} \sin\alpha}$.

13. An indefinitely great number of indefinitely thin cylindrical shells, just fitting one within another, are revolving with different angular velocities, but in the same sense, about their common axis; also the angular velocity of each shell is proportional to a positive power (the n^{th}) of its radius, and that of the outermost shell is ω . Prove that if the system of shells

be suddenly united into a solid cylinder, the cylinder will revolve about its axis with the angular velocity $\frac{4\omega}{n+4}$.

If the bodies composing the Solar system were suddenly to become rigidly connected, explain what the nature of the subsequent motion would be.

If ω_r be the angular velocity about the axis of any particle, at a distance r from the axis, the area described by it in the time t

$$= \frac{1}{2}r^2\omega_r \cdot t = \frac{1}{2}r^2\omega \frac{r^n}{a^n} \cdot t = \frac{1}{2}\omega \frac{r^{n+2}}{a^n} \cdot t,$$

if a be the radius of the outer shell.

Hence the sum of the areas described by all the particles in the same cylindrical shell, of thickness δr ,

$$= \pi\omega \frac{r^{n+3}}{a^n} t \cdot \delta r.$$

Hence the sum of all areas described by all particles in time t

$$\begin{aligned} &= \pi\omega \frac{t}{a^n} \int_0^a r^{n+3} \delta r \\ &= \pi\omega t \frac{a^4}{n+4}. \end{aligned}$$

If ϖ be the angular velocity after uniting, the sum of the areas described by all the areas in an equal time

$$\begin{aligned} &= \pi\varpi t \int_0^a r^3 \delta r \\ &= \pi\varpi t \cdot \frac{a^4}{4}. \end{aligned}$$

By the principle of the conservation of areas, the two above sums of areas must be equal, or

$$\pi\varpi t \frac{a^4}{4} = \pi\omega t \frac{a^4}{n+4};$$

$$\therefore \varpi = \frac{4\omega}{n+4}.$$

In the Solar system the areas of the orbits vary as the square of the mean distances, and the periodic times squared as the mean distances cubed, and therefore the mean angular velocities vary as the square roots of the mean distances. Hence, if all the bodies of the system were suddenly to become rigidly connected, the bodies nearer the sun would have their motions suddenly accelerated, and those furthest from it would be suddenly retarded; after which all would proceed with a common uniform and, so to speak, an average angular motion.

1850.

1. A parallelogram, whose centre is fixed, is rotating about one of its principal axes in its plane; find how it must be struck that, after the blow, it may rotate with the same angular velocity about the other.

Since the effect of a blow upon the velocity of rotation of the body about a principal-axis depends only upon the moment of the blow about that axis, it is plain that the blow must in this case be perpendicular to the plane of the parallelogram, and its moments about the two principal axes in its plane must be equal and be due to the velocity of rotation already existing about one of them, and in a direction to destroy it.

Let A, B , be the moments of inertia about these principal axes, ω the velocity of rotation about A before the blow (I) is given, \bar{x}, \bar{y} , the coordinates of the point of application of the impulse;

$$\therefore \bar{x} = \frac{A\omega}{I},$$

$$\bar{y} = \frac{B\omega}{I} :$$

which equations determine the point of application of the impulse.

2. Three equal smooth spheres (radius r) are placed together on a horizontal plane, and kept in contact by a string passed round them in the plane of their centres. A cone of given weight (W) and vertical angle (2α), is placed between them

so that its axis is vertical: find the tension of the string; and if the string be suddenly cut, find when the cone will strike the plane.

If R be the pressure between any sphere and the cone, we have for the equilibrium of the cone

$$3R \sin \alpha = W;$$

and for that of any sphere,

$$\begin{aligned} R \cos \alpha &= 2T \cos \frac{1}{6}\pi \\ &= 3^{\frac{1}{2}}.T, \\ \text{or } T &= \frac{W \cot \alpha}{3.3^{\frac{1}{2}}}, \end{aligned}$$

the required tension.

At the time t after the string is cut, let y be the height of the vertex of the cone above the plane, x the distance of any sphere from the axis of the cone: the equation of *vis viva* gives us, if W' be the weight of any sphere,

$$W \left(\frac{dy}{dt} \right)^2 + 3W' \left(\frac{dx}{dt} \right)^2 = 2Wg(y_0 - y),$$

where y_0 equals the height of the vertex of the cone in the position of equilibrium.

We have also to express the condition, that the motion of the point of any sphere in contact with the cone in the direction perpendicular to the generating line of the cone through the point of its contact with that sphere, is equal to the motion of that point in the same direction: or

$$\begin{aligned} \frac{dx}{dt} \cos \alpha &= - \frac{dy}{dt} \sin \alpha; \\ \therefore \frac{dx}{dt} &= - \frac{dy}{dt} \tan \alpha. \end{aligned}$$

Hence the above equation becomes

$$(W + 3W' \tan^2 \alpha) \left(\frac{dy}{dt} \right)^2 = 2Wg(y_0 - y):$$

or differentiating,

$$\frac{d^2y}{dt^2} = - \frac{Wg}{W + 3W' \tan^2 \alpha} = -f \text{ suppose,}$$

a constant retarding force. And the cone starts from rest; therefore the time of describing the space y_0 ,

$$T = \left(\frac{2f}{y_0} \right)^{\frac{1}{2}}.$$

To find y_0 . From fig. 107 it is evident that ON , the perpendicular on the axis of the cone from O , the centre of any sphere in the position of equilibrium, is the distance of any angular point of any equilateral triangle of side $2r$ from its centre;

$$\therefore ON = r \sec \frac{\pi}{6} = \frac{2}{3^{\frac{1}{2}}} r;$$

$$\therefore OT = ON \sec \alpha = \frac{2}{3^{\frac{1}{2}}} r \sec \alpha,$$

$$\text{and } PT = \left(\frac{2}{3^{\frac{1}{2}}} \sec \alpha - 1 \right) r;$$

$$\therefore CT = PT \operatorname{cosec} \alpha$$

$$= \left(\frac{2}{3^{\frac{1}{2}}} \sec \alpha \operatorname{cosec} \alpha - \operatorname{cosec} \alpha \right) r;$$

$$\therefore CN = CT - NT = CT - OT \sin \alpha$$

$$= \left\{ \frac{2}{3^{\frac{1}{2}}} (\sec \alpha \operatorname{cosec} \alpha - \tan \alpha) - \operatorname{cosec} \alpha \right\} r$$

$$= \left(\frac{2}{3^{\frac{1}{2}}} \cot \alpha - \operatorname{cosec} \alpha \right) r,$$

$$\text{and } y_0 = r - CN$$

$$= \left(1 + \operatorname{cosec} \alpha - \frac{2}{3^{\frac{1}{2}}} \cot \alpha \right) r;$$

$$\therefore T = \left\{ \frac{2Wg}{\left(1 + \operatorname{cosec} \alpha - \frac{2}{3^{\frac{1}{2}}} \cot \alpha \right) (W + 3W' \tan^2 \alpha)} \right\}^{\frac{1}{2}}.$$

3. Two equal particles of mass m are fixed at the extremities of the axis of a prolate spheroid, of which the mass is M , the eccentricity of the generating ellipse being e . The spheroid is struck by a couple and then left to move freely;

shew that throughout the motion it will constantly have contact with a single plane, if

$$m = \frac{1}{10}Me^2.$$

The spheroid will manifestly have contact with a fixed plane parallel to the invariable plane, if it be similar to the momental spheroid of the system consisting of the spheroid with the two masses (m) at its poles. Let A, B , be the moments of inertia of this system about its axis of figure and an axis through its centre perpendicular to its axis of figure. Hence, if a, b , be the semiaxes of the generating ellipse, the condition of contact with a single plane is

$$Aa^2 = Bb^2 \dots\dots\dots(1).$$

$$\text{Now} \quad A = \frac{2}{5}Mb^2, \quad B = \frac{1}{5}M(a^2 + b^2) + 2ma^2,$$

and condition (1) becomes

$$\frac{2}{5}Ma^2b^2 = \frac{1}{5}M(a^2 + b^2)b^2 + 2ma^2b^2,$$

$$\text{or} \quad 2ma^2 = \frac{1}{5}M(a^2 - b^2);$$

$$\therefore m = \frac{1}{10}M \frac{a^2 - b^2}{a^2} \\ = \frac{1}{10}Me^2.$$

4. A small arc of a hoop is removed and replaced by two small straight lines, tangents to the circle at the ends of the arc, their mass being so disposed that the centre of gravity remains still at the centre of the hoop. If the hoop be now rolled along a horizontal plane, sufficiently rough to prevent sliding, with an angular velocity ω not great enough to make it leap, shew that motion will never cease unless

$$\frac{\log \left\{ \frac{\omega^2}{2ga} \cdot \frac{a^2 + k^2 \cos^2 \alpha}{(1 - \cos \alpha) \cos \alpha} \right\}}{2 \log \left(1 + \frac{a^2}{a^2 + k^2} \tan^2 \alpha \right)}$$

be a whole number; where a is the radius of the hoop, k its radius of gyration, and 2α the angle subtended by the arc removed.

The motion of rolling on the circular rim will be uniform: hence the angular velocity at the time of impinging on the apex of the tangent will be ω . To determine the motion immediately after this impact. See fig. 108.

Let F, F' be the impulsive actions between the apex and the plane along the plane and perpendicular to it; v_x, v_y the velocities of the centre of gravity in the same directions, and ϖ the angular velocity after impact. The equations for finding v_x, v_y , and ϖ , are

$$v_x = a\omega - \frac{F}{M},$$

$$v_y = \frac{F'}{M},$$

$$Mk^2\varpi = Mk^2\omega + Fa - F'a \tan \alpha.$$

Also, to express the condition that the apex must be at rest after impact, we have

$$v_x - a\varpi = 0,$$

$$v_y - a \tan \alpha \cdot \varpi = 0.$$

Hence we have

$$k^2\varpi = k^2\omega + a^2(\omega - \varpi) - (a \tan \alpha)^2 \cdot \varpi,$$

$$\text{or } \varpi = \frac{a^2 + k^2}{a^2(1 + \tan^2 \alpha) + k^2} \cdot \omega$$

$$= n\omega \text{ suppose.}$$

Next, to consider the continuous motion of turning about the apex: let θ be the inclination to the horizon of the radius to the apex at the interval t after impact. Then, taking the equation of moments about the apex,

$$M(k^2 + a^2 \sec^2 \alpha) \frac{d^2 \theta}{dt^2} = -Mga \sec \alpha \cos \theta;$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = -\frac{2ga \sec \alpha}{k^2 + a^2 \sec^2 \alpha} \sin \theta + C$$

$$= \varpi^2 - \frac{2ga \sec \alpha}{k^2 + a^2 \sec^2 \alpha} (\sin \theta - \cos \alpha).$$

When $\theta = \frac{1}{2}\pi$, $\left(\frac{d\theta}{dt}\right)^2 = \omega^2 - \frac{2ga \sec \alpha}{k^2 + a^2 \sec^2 \alpha} (1 - \cos \alpha) \dots (1);$
 $\theta = \frac{1}{2}\pi + \alpha$, $\left(\frac{d\theta}{dt}\right)^2 = \omega^2.$

In passing from a motion of rotation about the axis to a motion of rolling on the circular rim, there is no impulsive motion; hence the angular motion at the time of the second impact of the apex on the plane is ω or $n\omega$. Similarly, the angular velocity at the time of the n^{th} impact is $n^r \omega$.

Now, if the hoop ever comes to rest it must be by just balancing on the apex: suppose this happens when it is rolling over the apex for the m^{th} time; then equation (1) shews that we must have

$$0 = n^{2m} \omega^2 - \frac{2ga \sec \alpha}{k^2 + a^2 \sec^2 \alpha} (1 - \cos \alpha),$$

$$\text{or } 2m \log n = \log \left\{ \frac{2ga \sec \alpha}{(k^2 + a^2 \sec^2 \alpha) \omega^2} (1 - \cos \alpha) \right\},$$

$$\text{or } m = \frac{\log \left\{ \frac{\omega^2}{2ga} \cdot \frac{(a^2 + k^2 \cos^2 \alpha)}{(1 - \cos \alpha) \cos \alpha} \right\}}{2 \log \left\{ 1 + \frac{a^2}{a^2 + k^2} \tan^2 \alpha \right\}},$$

by inverting both the quantities under the logarithmic sign. Hence, if this expression for m be a whole number, the hoop will come to rest as it is rolling over the apex for the m^{th} time: if it be not a whole number it will never come to rest.

5. Two similar homogeneous cords are similarly stretched, and one of them loaded at its middle point with a small weight μ ; shew that the fundamental note of the loaded cord will be lower than that of the other, and that if t denote the time of vibration of the loaded cord for any possible note, the values of t are given by the equation

$$\tan \left\{ \frac{\pi l}{t} \left(\frac{\varepsilon}{\tau} \right)^{\frac{1}{2}} \right\} = \frac{t}{\pi \mu} (\varepsilon \tau)^{\frac{1}{2}},$$

where l is the length of the cord, ε the mass of a unit of length,

and τ its tension. Under what initial circumstances will both strings sound the same note?

See Duhamel in the *Journ. de l'Ecole Polytech.* tom. xvii.

It is evident that when the loaded cord is sounding its fundamental note, its two halves meet in a salient angle pointing from the line joining its extremities. Thus the two halves are portions of the trochoid which a longer cord would assume in vibrating, and will vibrate in the same manner as if they actually were parts of such a trochoid: hence the loaded cord is virtually longer than the unloaded one, and capable of a deeper note.

Let x, y , be the coordinates of any particle of the string at the time t , referred to the line joining the extremities of the string, and a line through its middle point perpendicular to it and in the plane of vibration, as axes of x and y . The equation of transversal vibrations will be (Poisson, *Mécanique*, n°. 490)

$$y = h \sin \frac{m\pi}{\lambda} \left(\frac{1}{2}l - x \right) \cos \frac{m\pi}{\lambda} at,$$

where h is the greatest value of y for any particle, $a = \left(\frac{\tau}{\varepsilon} \right)^{\frac{1}{2}}$, and λ a quantity to be determined by the circumstance that the middle point of the string is attached to the weight μ .

Let y' be the ordinate of this weight at the same time t : the equation of motion of μ is

$$\begin{aligned} \mu \frac{d^2 y'}{dt^2} &= 2\tau \left(\frac{dy}{dx} \right)_{x=0} \\ &= -2\tau \cdot \frac{hm\pi}{\lambda} \cos \frac{m\pi}{\lambda} \frac{l}{2} \cos \frac{m\pi}{\lambda} at; \\ \therefore y' &= \frac{2\tau}{\mu} \cdot \frac{h\lambda}{m\pi} \cdot \frac{1}{a^2} \cos \frac{m\pi}{\lambda} \frac{l}{2} \cos \frac{m\pi}{\lambda} at. \end{aligned}$$

Now $y' = y_{x=0}$

$$= h \sin \frac{m\pi}{\lambda} \frac{l}{2} \cos \frac{m\pi}{\lambda} at;$$

$$\begin{aligned} \therefore \tan \frac{m\pi}{\lambda} \frac{l}{2} &= \frac{2\tau}{\mu} \cdot \frac{\lambda}{m\pi} \cdot \frac{1}{a^2} \\ &= \frac{2\varepsilon}{\mu} \cdot \frac{\lambda}{m\pi}, \quad \text{since } a^2 = \frac{\tau}{\varepsilon}. \end{aligned}$$

Now, let t represent the time of vibration ;

$$\therefore t = \frac{2\lambda}{m} \cdot \frac{1}{a} = \frac{2\lambda}{m} \cdot \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{2}},$$

and the above equation becomes, by eliminating λ ,

$$\tan \left\{ \frac{\pi l}{t} \cdot \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{2}} \right\} = \frac{t}{\pi \mu} (\varepsilon \tau)^{\frac{1}{2}},$$

the required equation for the determination of all possible values of t .

The value of t answering to the fundamental note is its greatest value; it is plainly such that

$$\frac{\pi l}{t} \cdot \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{2}} < \frac{\pi}{2}.$$

Now suppose μ indefinitely small, or the weight removed, the value t' of t then answering to the fundamental note is

$$\frac{\pi l}{t'} \cdot \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{2}} = \frac{\pi}{2},$$

and therefore $t > t'$, or the fundamental note is lower for the loaded than for the unloaded cord, as shewn above.

The two cords will evidently sound the same note when the middle point of each is made a node; in which case the note will be that due to a length which is any submultiple of $\frac{1}{2}l$.

6. If a body hang by a string, and through any point of the string a series of horizontal lines be drawn, with any one of which the body may be rigidly connected and perform small oscillations about it, the time of oscillation will be a maximum about a line at right angles to the one about which it is a minimum: prove this, and shew how to find the position of these two lines, and the time of oscillation about any other, in terms of the times about these two and the angle which it makes with them.

If t be the time of the body's oscillation about any one of these lines about which its moment of inertia is Q , we have the relation

$$t = 2\pi \left(\frac{Q}{Mhg} \right)^{\frac{1}{2}},$$

where M is the body's mass, h the depth of its centre of gravity below the horizontal line in question. Now the general expression for Q is

$$Q = \Sigma \delta m (x^2 + y^2 + z^2);$$

therefore if we make the horizontal plane through the lines the plane of xy , $\Sigma \delta m z^2$ is the same for all the lines, and the relative magnitudes of Q for the different lines will depend upon

$$\Sigma \delta m (x^2 + y^2).$$

But this expression will be unaltered if we project every particle of the body upon the plane of xy . Hence, as in the case of a plane lamina, we shall have two axes in the plane at right angles to each other, for one of which Q will be a maximum and for the other a minimum: hence the first part of the proposition is true.

Also, as far as the part $\Sigma \delta m (x^2 + y^2)$ of Q is concerned, we shall have the usual relation

$$Q = Q_1 \cos^2 \theta + Q_2 \sin^2 \theta,$$

where Q_1 , Q_2 are the maximum and minimum moments, and θ the angle which the axis of Q makes with that of Q_1 : hence the true relation between Q , Q_1 , and Q_2 is

$$Q - \Sigma \delta m . z^2 = (Q_1 - \Sigma \delta m z^2) \cos^2 \theta + (Q_2 - \Sigma \delta m z^2) \sin^2 \theta,$$

$$\text{or } Q = Q_1 \cos^2 \theta + Q_2 \sin^2 \theta, \text{ as before.}$$

Hence, if t , t_1 , t_2 , be the times about the lines about which Q , Q_1 , Q_2 , are the moments of inertia,

$$t = (t_1^2 \cos^2 \theta + t_2^2 \sin^2 \theta)^{\frac{1}{2}}.$$

To find the positions of the lines of greatest and least moments in the given horizontal plane.

Let the direction-cosines of this plane referred to the principal axes through the point where the string pierces the plane be l , m , n ; and let Q be the moment about the line whose direction-cosines referred to the same axes are α , β , γ ; then we are to have, if A , B , C , be the principal moments,

$$Q = A\alpha^2 + B\beta^2 + C\gamma^2 = \text{a maximum or minimum}$$

subject to the conditions

$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

$$\text{and } l\alpha + m\beta + n\gamma = 0,$$

which last is the equation to the horizontal plane.

Differentiating these three equations with respect to α , β , and γ , we find

$$A\alpha d\alpha + B\beta d\beta + C\gamma d\gamma = 0 \dots\dots\dots (1),$$

$$\alpha d\alpha + \beta d\beta + \gamma d\gamma = 0 \dots\dots\dots (2),$$

$$\text{and } l d\alpha + m d\beta + n d\gamma = 0 \dots\dots\dots (3).$$

Using the arbitrary multipliers λ and μ , we deduce the equations

$$A\alpha + \lambda\alpha = \mu l \dots\dots\dots (4),$$

$$B\beta + \lambda\beta = \mu m \dots\dots\dots (5),$$

$$C\gamma + \lambda\gamma = \mu n \dots\dots\dots (6).$$

(4) α + (5) β + (6) γ gives

$$\left. \begin{aligned} Q + \lambda &= 0; \\ \therefore (A - Q)\alpha &= \mu l \\ (B - Q)\beta &= \mu m \\ (C - Q)\gamma &= \mu n \end{aligned} \right\} \dots\dots\dots (A).$$

Hence $\alpha^2 + \beta^2 + \gamma^2 = 1 = \mu^2 \left\{ \frac{l^2}{(A - Q)^2} + \frac{m^2}{(B - Q)^2} + \frac{n^2}{(C - Q)^2} \right\} \dots (7),$

and

$$\frac{1}{\mu} (l\alpha + m\beta + n\gamma) = 0 = \frac{l^2}{A - Q} + \frac{m^2}{B - Q} + \frac{n^2}{C - Q},$$

which gives a quadratic for the determination of Q_1 and Q_2 . The substitution of the value of μ from equation (7) in equations (A) will give us the values of α , β , γ , and so determine the positions of the lines of maximum and minimum moments.

1851.

1. The locus of an axis passing through a fixed point of a solid body, and such that the moment of inertia round it of the body is constant, is a cone of the second order, and the cones

corresponding to different values of the constant moment have the same directions of circular sections.

The expression for Q , the moment of inertia about an axis whose direction-cosines referred to the principal axes are α, β, γ , in terms of the principal moments A, B, C , is

$$Q = A\alpha^2 + B\beta^2 + C\gamma^2;$$

or if the axes of A, B, C , are those of x, y , and z ,

$$Q(x^2 + y^2 + z^2) = Ax^2 + By^2 + Cz^2,$$

the equation to a cone of the second order.

Let A, B, Q, C , be in descending order of magnitude, the above equation may be put in the form

$$(A - Q)x^2 + (B - Q)y^2 - (Q - C)z^2 = 0;$$

$$\text{and if } z = mx + ny + c$$

be the equation to any plane which cuts the cone in a circle, we have (*Gregory's Solid Geometry*, Art. 124)

$$n = 0,$$

$$m = \pm \left\{ \frac{A - Q - (B - Q)}{B - Q + (Q - C)} \right\}^{\frac{1}{2}}$$

$$= \pm \left(\frac{A - B}{B - C} \right)^{\frac{1}{2}},$$

which shews the direction of the circular sections to be independent of Q .

2. Determine the motion of a heavy solid composed of two equal right cones placed together base to base, and which rolls without sliding upon two intersecting lines inclined at equal angles to the vertical, the common base of the cones moving in the plane which bisects the angles between the vertical planes through the lines.

We shall apply the principle of *vis viva*.

Let CA, CB (fig. 109) be the two lines; and at the time t let them touch the two cones in P, P' : let GM the height of G

the centre of gravity of the solid above the horizontal plane through $C = z$, $CM = x$. Then if r be the distance of P from the axis of the cone, and δ the inclination of the plane ACB to the horizon, PN the height of $P = z - r \cos \delta$, or if $CP = \rho$,

$$\rho \sin \gamma = z - r \cos \delta;$$

$$\therefore z = \rho \sin \gamma + r \cos \delta \dots \dots \dots (1).$$

Also, if $\angle MCN = \varepsilon$,

$$x = CN \cos \varepsilon = \rho \cos \gamma \cos \varepsilon \dots \dots \dots (2).$$

Now as the cone rolls along, the locus of P on the cone will be a curve like the dotted curve in the figure: let δs be an element of the length of this curve answering to the rotation of the solid through a small angle $\delta \theta$, then

$$\delta s = \delta (r \operatorname{cosec} \alpha) \operatorname{cosec} \beta \dots \dots \dots (3),$$

if β be the inclination of either rod to the common base of the cones, or 2β the inclination of the rods to each other: also

$$\delta (r \operatorname{cosec} \alpha) = \delta (r \theta) \tan \beta \dots \dots \dots (4);$$

$$\therefore \delta \rho = -\delta s = -\delta r \operatorname{cosec} \alpha \operatorname{cosec} \beta \text{ by (3);}$$

$$\therefore \rho = (r_0 - r) \operatorname{cosec} \alpha \operatorname{cosec} \beta,$$

if r_0 is the radius of the common base of the cones: also by (4),

$$r - r_0 = r \theta \tan \beta \sin \alpha,$$

if $\theta = 0$ when the solid touches both the lines at C ;

$$\therefore \theta = \left(1 - \frac{r_0}{r}\right) \cot \beta \operatorname{cosec} \alpha.$$

Hence, by (1) and (2),

$$z = (r_0 - r) \operatorname{cosec} \alpha \operatorname{cosec} \beta \sin \gamma + r \cos \delta \dots \dots \dots (5),$$

$$x = (r_0 - r) \operatorname{cosec} \alpha \operatorname{cosec} \beta \cos \gamma \cos \varepsilon.$$

Now the equation of *vis viva* gives us

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + k^2 \left(\frac{d\theta}{dt}\right)^2 = 2g(z_0 - z),$$

where z_0 is the height of ρ where the solid starts from rest ;

$$\begin{aligned} \therefore \{ & \operatorname{cosec}^2 \alpha \operatorname{cosec}^2 \beta \cos^2 \gamma \cos^2 \varepsilon + (\operatorname{cosec} \alpha \operatorname{cosec} \beta \sin \gamma - \cos \delta)^2 \\ & + \frac{h^2 r_0^2}{r^4} \cot^2 \beta \operatorname{cosec} \alpha \} \left(\frac{dr}{dt} \right)^2 \\ & = 2g \{ z_0 - r_0 \operatorname{cosec} \alpha \operatorname{cosec} \beta \sin \gamma \\ & \quad + r (\operatorname{cosec} \alpha \operatorname{cosec} \beta \sin \gamma - \cos \delta) \}, \end{aligned}$$

an equation which, when integrated, will give us r , and therefore also x and z , at any time t .

From equation (5) it appears that as z will necessarily be diminished by the force of gravity, if the solid starts from rest it will roll toward C or from it, according as

$$\operatorname{cosec} \alpha \operatorname{cosec} \beta \sin \gamma > \text{ or } < \cos \delta.$$

3. Shew that the difference of the moments of inertia of a body round two axes in a given plane which are equally inclined to a fixed line in the same plane, is proportional to the sine of the angle between those axes.

Let Q_1, Q_2 be the maximum and minimum moments about lines in that plane, Q, Q' the moments about any two lines in the plane making angle θ, θ' with the axis of the moment Q_1 ; then, by Problem 6, 1850,

$$\begin{aligned} Q &= Q_1 \cos^2 \theta + Q_2 \sin^2 \theta, \\ Q' &= Q_1 \cos^2 \theta' + Q_2 \sin^2 \theta'; \\ \therefore Q - Q' &= Q_1 (\cos^2 \theta - \cos^2 \theta') + Q_2 (\sin^2 \theta - \sin^2 \theta') \\ &= Q_1 \sin(\theta' - \theta) \sin(\theta + \theta') + Q_2 \sin(\theta - \theta') \sin(\theta + \theta') \\ &= (Q_1 - Q_2) \sin(\theta + \theta') \sin(\theta' - \theta) \\ &= (Q_1 - Q_2) \sin 2\alpha \sin(\theta' - \theta), \end{aligned}$$

where α is the angle the fixed line makes with the axis of Q ,

$$\propto \sin(\theta' - \theta),$$

$$\propto \text{sine of the angle between the axes of } Q \text{ and } Q'.$$

HYDROSTATICS.

1849.

1. A plane body, one of the edges of which is a straight line, is immersed in water so as to have this straight line coincident with the surface; shew how the depth of the centre of pressure may be deduced from observation of the time of a small oscillation in vacuum of the body about its rectilinear side.

When the body is immersed with its plane vertical,

let z' be the depth of the centre of pressure below the surface,

... \bar{z} gravity,

... z any point of the body

$$\begin{aligned}\text{Then } z' &= \frac{\int g \rho z^2 dz}{\int g \rho z dz} \\ &= \frac{M k'^2}{M \bar{z}} = \frac{k'^2}{\bar{z}},\end{aligned}$$

(where k' is the radius of gyration about the straight edge)

$$= l,$$

the length of the simple pendulum when the body makes small oscillations about the rectilinear side. Hence, if T be the time of small oscillations,

$$T = 2\pi \left(\frac{l}{g} \right)^{\frac{1}{2}};$$

$$\therefore z' = l = g \frac{T^2}{4\pi^2},$$

the formula for the determination of z' from T .

2. A cylinder the radius of which is a , having its axis vertical and containing incompressible fluid (density ρ), revolves about its axis with an angular velocity $\omega = \left(n \frac{g}{a} \right)^{\frac{1}{2}}$,

n being > 1 ; a sphere (density ρ'), whose radius is also a , on being put into the cylinder, is supported in such a position that it touches the free surface at its vertex: shew that

$$\frac{\rho'}{\rho} = \left(1 - \frac{1}{n}\right)^3.$$

The equation of equilibrium is

$$\text{volume of fluid displaced} = \frac{\rho'}{\rho} \cdot \frac{4}{3}\pi a^3 \dots\dots\dots(1).$$

Fig. 110 shews the position of the sphere in the fluid, the dotted line representing the continuation of the section by the plane of the paper of the free surface.

Let V , the vertex of the free surface, be the origin of coordinates, the axis of the cylinder that of z , and r the distance of any point from it. Then the equations to the free surface and that of the sphere are

$$r^2 = \frac{2}{\omega^2}gz = \frac{2a}{n}z,$$

$$\text{and } r^2 = 2az - z^2;$$

therefore if z' be the height above V of the circle of intersection,

$$0 = 2a\left(1 - \frac{1}{n}\right)z' - z'^2,$$

$$\text{or } z' = 2a\left(1 - \frac{1}{n}\right).$$

Hence the volume of the fluid displaced

$$\begin{aligned} &= \pi \int_0^{z'} (2az - z^2) dz - \frac{\pi}{2} \cdot \frac{2a}{n} z'^2 \\ &= \pi \left\{ a\left(1 - \frac{1}{n}\right)z'^2 - \frac{1}{3}z'^3 \right\} \\ &= 4\pi a^2 \left(1 - \frac{1}{n}\right)^2 \left\{ a\left(1 - \frac{1}{n}\right) - \frac{2}{3}a\left(1 - \frac{1}{n}\right) \right\} \\ &= \frac{4}{3}\pi a^3 \left(1 - \frac{1}{n}\right)^3; \end{aligned}$$

therefore from (1)

$$\frac{\rho'}{\rho} \cdot \frac{4}{3} \pi a^3 = \frac{4}{3} \pi a^3 \left(1 - \frac{1}{n}\right)^3,$$

$$\text{or } \frac{\rho'}{\rho} = \left(1 - \frac{1}{n}\right)^3.$$

3. If X, Y, Z , be the forces acting at a point (xyz) of a mass of heterogeneous fluid in equilibrium, and $Xdx + Ydy + Zdz$, be not a perfect differential, then the pressure and density will be constant throughout the curves of which the differential equations are

$$\frac{\frac{dx}{dY} - \frac{dZ}{dy}}{\frac{dZ}{dx} - \frac{dY}{dz}} = \frac{\frac{dy}{dZ} - \frac{dX}{dz}}{\frac{dX}{dy} - \frac{dY}{dx}}.$$

Let p, ρ be the pressure and density at the point (xyz) ; $p + dp$ the pressure at the point $(x + dx, y + dy, z + dz)$, then

$$dp = \rho (Xdx + Ydy + Zdz) \dots \dots \dots (1).$$

In order that this equation may hold, and therefore equilibrium be possible, we must have the right-hand side of this equation a perfect derivative of three independent variables, or we must have

$$\frac{d\rho Y}{dz} = \frac{d\rho Z}{dy},$$

$$\frac{d\rho Z}{dx} = \frac{d\rho X}{dz},$$

$$\frac{d\rho X}{dy} = \frac{d\rho Y}{dx};$$

$$\text{or } \left. \begin{aligned} \rho \left(\frac{dY}{dz} - \frac{dZ}{dy} \right) &= Z \frac{d\rho}{dy} - Y \frac{d\rho}{dz} \\ \rho \left(\frac{dZ}{dx} - \frac{dX}{dz} \right) &= X \frac{d\rho}{dz} - Z \frac{d\rho}{dx} \\ \rho \left(\frac{dX}{dy} - \frac{dY}{dx} \right) &= Y \frac{d\rho}{dx} - X \frac{d\rho}{dy} \end{aligned} \right\} \dots \dots \dots (\Lambda).$$

Multiplying these equations in order by X , Y , Z , and adding, we get

$$X\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) + Y\left(\frac{dZ}{dx} - \frac{dX}{dz}\right) + Z\left(\frac{dX}{dy} - \frac{dY}{dx}\right) = 0 \dots (2),$$

as the condition which the forces X , Y , Z , must satisfy in order that they may be able to produce equilibrium.

Now let dx , dy , dz , in the expression (1) for dp , be such that

$$\frac{dx}{\frac{dY}{dz} - \frac{dZ}{dy}} = \frac{dy}{\frac{dZ}{dx} - \frac{dX}{dz}} = \frac{dz}{\frac{dX}{dy} - \frac{dY}{dx}} \dots \dots \dots (3),$$

then dp will be the variation of p as we pass from one point to the adjacent point of any of the curves of which these are the differential equations. Now, combining (1) and (3), we get by (2)

$$dp = 0,$$

wherefore p is constant along the curves whose differential equations are (3).

Also from equations (A) we may put equation (3) in the form

$$\frac{dx}{Z \frac{d\rho}{dy} - Y \frac{d\rho}{dz}} = \frac{dy}{X \frac{d\rho}{dz} - Z \frac{d\rho}{dx}} = \frac{dz}{Y \frac{d\rho}{dx} - X \frac{d\rho}{dy}} = r \text{ suppose ;}$$

$$\therefore dx = r \left(Z \frac{d\rho}{dy} - Y \frac{d\rho}{dz} \right),$$

$$dy = r \left(X \frac{d\rho}{dz} - Z \frac{d\rho}{dx} \right),$$

$$dz = r \left(Y \frac{d\rho}{dx} - X \frac{d\rho}{dy} \right) :$$

and multiplying these equations by $\frac{d\rho}{dx}$, $\frac{d\rho}{dy}$, $\frac{d\rho}{dz}$, respectively, we get

$$d\rho = \frac{d\rho}{dx} dx + \frac{d\rho}{dy} dy + \frac{d\rho}{dz} dz = 0 ;$$

or ρ is also constant along these curves.

1850.

1. Three equal cylinders are placed in contact upon a horizontal plane, sufficiently rough to prevent sliding: find how much water must be poured into the space between the cylinders, in order to disturb the equilibrium.

Let h be the depth of the water poured in when each cylinder is on the point of turning about a tangent line to its base, in which case the water will run out between the cylinders.

Now the moment of the fluid pressures upon each cylinder about the tangent line to the base about which the cylinder would begin to turn, is the same as the moment of the fluid pressures on a vertical rectangle of height h and breadth equal to the radius (r) of each cylinder about its base*

$$\begin{aligned} &= r \int_0^h g\rho (h-z) z dz \\ &= r \cdot g\rho h^3 \left(\frac{1}{2} - \frac{1}{3}\right) \\ &= \frac{1}{6} g\rho h^3 r. \end{aligned}$$

Now this must equal the moment of the weight (Mg) of each cylinder about the same line or Mgr ,

$$\begin{aligned} \therefore \frac{1}{6} g\rho h^3 r &= Mgr, \\ \therefore h &= \left(\frac{6M}{\rho}\right)^{\frac{1}{3}} \end{aligned}$$

is the required height.

2. All space being supposed filled with an elastic fluid whose volume at a given density is known, the particles of which are attracted to a given point by a force varying as the distance: find the pressure on a circular disc placed with its centre at the centre of force.

Let μ = absolute force of attraction at distance unity; the attractions X, Y, Z , parallel to the axes at the point (xyz) are

* For this is the nature of the section of each cylinder supposed of a height h , made by a plane through the lines of its contact with the other cylinders.

$-\mu x, -\mu y, -\mu z$, the centre of force being origin: hence

$$\begin{aligned} dp &= \rho(Xdx + Ydy + Zdz) \\ &= -\mu\rho(xdx + ydy + zdz) = -\mu\rho r dr \text{ if } r^2 = x^2 + y^2 + z^2; \\ &\text{and } \rho = kp; \\ \therefore \frac{dp}{p} &= -\mu k r dr; \\ \therefore p &= C\varepsilon^{-\frac{1}{2}\mu k r^2}. \end{aligned}$$

To determine C , we have

$$\begin{aligned} \rho &= kp = Ck\varepsilon^{-\frac{1}{2}\mu k r^2}; \\ \therefore \delta M &= \text{mass contained between two consecutive spheres having } C \text{ for centre, radii } r \text{ and } r + \delta r \\ &= 4\pi\rho r^2\delta r = 4\pi Ck\varepsilon^{-\frac{1}{2}\mu k r^2}\delta r; \\ \therefore M &= \text{whole mass, and therefore known,} \\ &= 4\pi Ck \int_0^\infty r^2\varepsilon^{-\frac{1}{2}\mu k r^2} dr. \end{aligned}$$

Let $\frac{1}{2}\mu k r^2 = z$,

$$\begin{aligned} \therefore r^3 &= \left(\frac{2}{\mu k}\right)^{\frac{3}{2}} z^{\frac{3}{2}}; \\ \text{and } r^2 dr &= \frac{2^{\frac{1}{2}}}{(\mu k)^{\frac{3}{2}}} z^{\frac{1}{2}} dz; \\ \therefore M &= 4\pi Ck \frac{2^{\frac{1}{2}}}{(\mu k)^{\frac{3}{2}}} \int_0^\infty z^{\frac{1}{2}}\varepsilon^{-z} dz; \\ \text{and } \int_0^\infty z^{\frac{1}{2}}\varepsilon^{-z} dz &= \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\ &= \frac{1}{2}\pi^{\frac{1}{2}}; \\ \therefore M &= C \frac{(2\pi)^{\frac{3}{2}}}{(\mu^{\frac{3}{2}}k)^{\frac{1}{2}}}, \end{aligned}$$

and M is known; hence C is also known.

Hence, if P be the pressure on the annulus (radius a) we have

$$\begin{aligned} \delta P &= 2\pi r \delta r.p \\ &= 2\pi C r \delta r \varepsilon^{-\frac{1}{2}\mu k r^2}; \end{aligned}$$

$$\begin{aligned}
 & \left. \begin{array}{l} \text{from } r = 0 \\ \text{to } r = a \end{array} \right\} \quad \therefore P = \frac{2\pi C}{\mu k} (C' - \epsilon^{-\frac{1}{2}\mu k r^2}) \\
 & \quad \quad \quad = \frac{2\pi C}{\mu k} (1 - \epsilon^{-\frac{1}{2}\mu k a^2}) \\
 & \quad \quad \quad = \left(\frac{\mu}{2\pi k} \right)^{\frac{1}{2}} M (1 - \epsilon^{-\frac{1}{2}\mu k a^2}).
 \end{aligned}$$

3. A hollow cylinder is filled with inelastic fluid and made to revolve about a vertical axis attached to the centre of its upper plane face with a velocity sufficient to retain it at the same inclination to the axis. Find at what point of the face a hole might be bored without loss of any fluid.

Let ω be the angular velocity of rotation: then, if the fluid were contained in an open vessel, the latus-rectum (l) of the generating parabola of the free surface would be $\frac{2g}{\omega^2}$. Now since it is supposed by the question that there is a point in the upper plane face where the pressure of the fluid is zero, it is manifest that the face touches the above free surface at this point. This point will evidently lie in the diameter of the face most inclined to the horizon, at a distance r suppose from the centre of the face. Let α be the inclination of the face to the vertical, h the distance of the vertex of the supposed free surface above the centre of the face, the equation to the free surface is

$$y^2 = l(x - h),$$

and for x, y we may write $r \cos \alpha, r \sin \alpha$,

$$\therefore r^2 \sin^2 \alpha = l(r \cos \alpha - h);$$

the roots of this equation are equal,

$$\begin{aligned}
 \therefore r &= \frac{1}{2} l \frac{\cos \alpha}{\sin^2 \alpha} \\
 &= \frac{g}{\omega^2} \cdot \frac{\cos \alpha}{\sin^2 \alpha}.
 \end{aligned}$$

4. A mass of inelastic fluid is contained between three co-ordinate planes, each of which attracts with a force which varies

as the distance, and the absolute forces of attraction μ_1, μ_2, μ_3 , are in harmonic progression. Half an ellipsoid is fixed with its plane surface against one of the coordinate planes, and its surface touching the other planes; its axes being parallel to the coordinate axes and proportional to $\mu_1^{-\frac{1}{2}}, \mu_2^{-\frac{1}{2}}, \mu_3^{-\frac{1}{2}}$. If there be not sufficient fluid quite to cover the ellipsoid, the uncovered part will be bounded by a semicircle.

The attractions X, Y, Z , parallel to the axes are $-\mu_1 x, -\mu_2 y, -\mu_3 z$;

$$\begin{aligned}\therefore dp &= Xdx + Ydy + Zdz \quad (\text{if } \rho = \text{unity}) \\ &= -(\mu_1 x dx + \mu_2 y dy + \mu_3 z dz); \end{aligned}$$

therefore the equation to the free surface is

$$\mu_1 x^2 + \mu_2 y^2 + \mu_3 z^2 = \text{a constant} = C \text{ suppose } \dots\dots (1).$$

The equation to the ellipsoid, if it be bisected by xz , is

$$\mu_1 (x-a)^2 + \mu_2 y^2 + \mu_3 (z-c)^2 = C' \dots\dots\dots (2).$$

(1) - (2) gives for the plane of intersection

$$2\mu_1 ax + 2\mu_3 cz = \text{a constant} = 2(C')^{\frac{1}{2}} A \text{ suppose};$$

$$\therefore \mu_1^{\frac{1}{2}} x + \mu_3^{\frac{1}{2}} z = A \dots\dots\dots (3),$$

$$\text{since } a = \left(\frac{C'}{\mu_1}\right)^{\frac{1}{2}}, \quad c = \left(\frac{C'}{\mu_3}\right)^{\frac{1}{2}}.$$

(3) may be put in the form

$$\mu_3 z^2 = A^2 - 2A\mu_1^{\frac{1}{2}} x + \mu_1 x^2;$$

subtracting this equation from (1) gives

$$2\mu_1 x^2 + \mu_2 y^2 = 2A\mu_1^{\frac{1}{2}} x - A^2 + C \dots\dots\dots (4),$$

the equation to the projection on (xy) of the curve of intersection.

Let ϕ equal the angle at which (3) is inclined to xy ;

$$\therefore \tan \phi = \left(\frac{\mu_1}{\mu_3}\right)^{\frac{1}{2}},$$

$$\text{and } \cos^2 \phi = \frac{\mu_3}{\mu_1 + \mu_3} = \frac{2\mu_2}{\mu_1},$$

since μ_1, μ_2, μ_3 , are in harmonic progression.

This equation, taken with (4), shews that the axes of the projection on xy of the curve of intersection parallel to x and y respectively, are in the ratio of $\cos \phi : 1$; hence the curve of intersection must be circular, evidently a semicircle, whose diameter lies in xz , and its plane perpendicular to xz .

5. A rectangular vessel is filled with fluid of twice its weight, and placed with its open end downwards upon a horizontal plane, which is then made to revolve round each side of the base successively, one of these sides being greater and the other less than three times its height: find when the fluid will begin to escape in each case, supposing the centres of gravity of the vessel and the fluid to coincide.

If the vessel had a base instead of being opened at the lower end, the moment of the fluid pressure on its inside about any side of the base would be the same as that of its weight acting at its centre of gravity: hence, when the vessel is open at the lower end, the moment of the fluid pressures about a side of the base will be that of the weight acting at its centre of gravity, minus the moment of the fluid pressures on the plane on which the vessel rests.

To find this moment, M suppose. Let the horizontal plane be supposed to have been turned through an angle α , and let r be the distance of any point in it from a horizontal line in it, at the same height as the highest edge of the vessel: the distance of the edge about which the vessel is being turned will be, if a be this edge, b the other edge, and h the height of the vessel, $b + h \cot \alpha$. Hence

$$\begin{aligned} M &= \int_{h \cot \alpha}^{b+h \cot \alpha} gpr \sin \alpha \cdot a dr (b + h \cot \alpha - r) \\ &= gpa \sin \alpha \int_{h \cot \alpha}^{b+h \cot \alpha} \{(b + h \cot \alpha) r - r^2\} dr \\ &= gpab \sin \alpha \left\{ \frac{1}{2} (b + h \cot \alpha) (b + 2h \cot \alpha) - \frac{1}{3} (b^2 + 3bh \cot \alpha + 3h^2 \cot^2 \alpha) \right\} \\ &= gpab \sin \alpha \left(\frac{1}{6} b^2 + \frac{1}{2} hb \cot \alpha \right). \end{aligned}$$

Let W be the weight of the vessel, and therefore $2W$ that of the fluid: then, when the water begins to flow out, the moment

about the edge a of all the forces on this vessel, including its weight, is zero, or

$$3W(\frac{1}{2}b \cos \alpha - \frac{1}{2}h \sin \alpha) - M = 0,$$

or, since $W = gpabh$, and $b = 3nh$ suppose,

$$\frac{3h}{2} (3nh \cos \alpha - h \sin \alpha) - 3nh \sin \alpha (\frac{1}{8} \cdot 3nh + \frac{1}{2}h \cot \alpha) = 0,$$

$$\text{or } 3n \cos \alpha - \sin \alpha - n(n \sin \alpha + \cos \alpha) = 0;$$

$$\therefore \tan \alpha = \frac{2n}{n^2 + 1};$$

which gives the value of α when the two values of n are substituted, one $>$, the other < 1 . In both cases, however, α is $< 45^\circ$.

HYDRODYNAMICS.

1848.

1. A cylindrical vessel, with its axis vertical, is filled with fluid, which issues from a great number of small orifices pierced in the side: find the surface which touches all the streams of spouting fluid.

This surface is evidently a surface of revolution, having the axis of the cylinder for axis. Its generating curve is the line which touches all the parabolic jets of water from the different orifices in the same generating line of the cylinder. These jets have all this generating line for axis, and a common directrix in the plane in which they lie, viz. the horizontal line at the level of the surface: for the velocity of efflux is that due to the distance from this line.

Hence, making the common axis and directrix axes of x and y respectively, the equation to the jet whose point of efflux is at a depth h is

$$y^2 = 4h(x - h).$$

To find the line which this curve always touches, differentiate with respect to h , considering x, y constant;

$$\therefore 0 = x - 2h,$$

and eliminating h ,

$$y^2 = 2x \cdot \frac{1}{2}x,$$

$$\text{or } y = x;$$

the equation a straight line through the origin, inclined to the vertical at an angle of 45° . Hence the surface required is a right-angled cone placed on the cylinder in an inverted position.

2. A closed vessel is filled with water, containing in it a piece of cork which is free to move; if the vessel be suddenly

moved forwards by a blow, shew that the cork will shoot forwards relatively to the water.

Suppose, for an instant, the cork removed, and its place occupied by solidified water; when the blow is struck this mass of solidified water will instantaneously receive a velocity V equal to that of the surrounding water, and the impulse on it will be MV , if M be its mass. But when the cork is in the place of this solidified water, the impulsive actions on it of the surrounding fluid will be the same as they were on the solidified water, and therefore the impulse on it will be the same. But the cork is lighter than the same volume of solidified water, and therefore the same impulse will impart a greater velocity to it, or the cork will move forward relatively to the water.

3. A closed vessel is filled with water which is at rest, and the vessel is then moved in any manner: apply the principle of the conservation of areas to prove that, if the vessel have any motion of rotation, no finite portion of the water can remain at rest relatively to the vessel.

The principle of conservation of areas about any axis must apply to the whole mass of water. But if any portion of the water remain at rest relatively to the vessel, we may suppose it to become solidified and rigidly attached to the vessel without altering the motion of any particle of the water: but in this case it is evident that the principle of the conservation of areas about any axis must also apply to the part of the water not solidified; consequently it must also apply to the solidified portion of the water which, since the water is originally at rest, can therefore have no motion of rotation, which is absurd if the vessel have any motion of rotation. Therefore, if the vessel have any motion of rotation there cannot be any finite portion of the water which remains at rest relatively to it.

1849.

1. Supposing the effect of friction in the case of aerial vibrations in a tube of uniform bore to be the production of

a retarding force on each particle equal to $f \times$ velocity, prove that the equation of motion will be satisfied by taking $c\varepsilon^{-\frac{1}{2}n} \sin \frac{2\pi}{\lambda} \left\{ a \left(1 - \frac{f^2 \lambda^2}{16a^2} \right)^{\frac{1}{2}} t - x \right\}$ as the type of the vibrations.

Let x be the coordinate of any particle at rest, $x + \xi$ its coordinate when displaced at time t ; then the equation of motion will be

$$\frac{d^2 \xi}{dt^2} = a^2 \frac{d^2 \xi}{dx^2} - f \frac{d\xi}{dt}.$$

Now, if we assume

$$\xi = c\varepsilon^{-\frac{1}{2}n} \sin \frac{2\pi}{\lambda} (nat - x), \quad \text{where } n^2 = 1 - \frac{f^2 \lambda^2}{16a^2 \pi^2},$$

we have

$$\begin{aligned} \frac{d\xi}{dt} + f\xi &= c\varepsilon^{-\frac{1}{2}n} \left\{ \frac{2\pi na}{\lambda} \cos \frac{2\pi}{\lambda} (nat - x) + \frac{f}{2} \sin \frac{2\pi}{\lambda} (nat - x) \right\}; \\ \therefore \frac{d^2 \xi}{dt^2} + f \frac{d\xi}{dt} &= -c\varepsilon^{-\frac{1}{2}n} \left\{ \frac{4\pi^2 n^2 a^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (nat - x) + \frac{f^2}{4} \sin \frac{2\pi}{\lambda} (nat - x) \right\} \\ &= -c\varepsilon^{-\frac{1}{2}n} \frac{4\pi^2 a^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (nat - x), \end{aligned}$$

by substitution of the value of n .

$$\begin{aligned} \text{Also} \quad a^2 \frac{d^2 \xi}{dx^2} &= -\frac{4\pi^2 a^2}{\lambda^2} c\varepsilon^{-\frac{1}{2}n} \sin \frac{2\pi}{\lambda} (nat - x); \\ \therefore \frac{d^2 \xi}{dt^2} + f \frac{d\xi}{dt} &= a^2 \frac{d^2 \xi}{dx^2} \\ \text{or} \quad \frac{d^2 \xi}{dt^2} &= a^2 \frac{d^2 \xi}{dx^2} - f \frac{d\xi}{dt}, \end{aligned}$$

and the equation of motion is satisfied.

2. Steam is rushing from a boiler through a conical pipe, the diameters of the extremities of which are D and d respectively: prove that if V and v be the corresponding velocities of the steam,

$$v = V \frac{D^2}{d^2} \varepsilon^{\frac{v^2 - V^2}{2k}},$$

where k is the pressure divided by the density, and supposed constant. The motion may be supposed to be that of a fluid diverging from a centre, the centre being the vertex of the cone, of which the pipe forms a portion.

Let p and ρ be the pressure and density at the distance r from the centre of motion at the time t , when the velocity at that point is u ; then, since the motion is wholly radial, its equation is

$$\frac{1}{\rho} \frac{dp}{dr} = - \frac{du}{dt} = - u \frac{du}{dr}, \text{ since the motion is steady ... (1).}$$

$$\text{Also } p = k\rho \dots \dots \dots (2).$$

The consideration of continuity gives the equation

$$upr^2 = \text{constant},$$

$$\text{or } upr^2 = \text{constant} \dots \dots \dots (3).$$

From (1) and (2),

$$\frac{k}{p} \frac{dp}{dr} = - u \frac{du}{dr},$$

$$\text{or } k \log p = C - \frac{1}{2}u^2.$$

Let P, p be the pressures at the two extremities of the pipe,

$$\therefore k \log \frac{P}{p} = \frac{1}{2}(v^2 - V^2),$$

$$\text{or } \frac{P}{p} = \varepsilon^{\frac{v^2 - V^2}{2k}}.$$

But from (3),

$$\frac{P}{p} = \frac{vd^2}{VD^2},$$

$$\therefore v = V \frac{D^2}{d^2} \varepsilon^{\frac{v^2 - V^2}{2k}}.$$

1851.

1. If a regular homogeneous tetrahedron be placed in any position whatever in a fluid whose density varies as the depth, shew that when the resistance of the fluid is neglected, the

tetrahedron will make vertical oscillations in the time $2\pi\left(\frac{h}{g}\right)^{\frac{1}{2}}$, h being the depth of the centre of gravity of the tetrahedron in the position of equilibrium.

We shall first shew that the centres of gravity of the tetrahedron and of the displaced fluid are in the same vertical, whatever position the tetrahedron occupies in the fluid.

Let the centre of gravity of the tetrahedron be at a depth z' below the surface, and be taken as origin of rectangular co-ordinates (xyz) , the latter vertically downwards. Let the density at a depth z below the surface be μz : the density at the point xyz will be $\mu(z' + z) = c + \mu z$ suppose.

Let \bar{x} , \bar{y} , \bar{z} , be the coordinates of the centre of gravity of the displaced fluid;

$$\therefore (\text{mass}) \bar{x} = \iiint (c + \mu z) x dx dy dz :$$

but

$$\iiint x dx dy dz = 0,$$

because the centre of gravity of the solid is origin, and

$$\iiint xz dx dy dz = 0,$$

because every system of rectangular axes through the centre of gravity of a regular solid is a *principal* system;

$$\therefore \bar{x} = 0, \quad \text{and similarly } \bar{y} = 0 :$$

hence the centre of gravity of the fluid displaced lies in the vertical line through that of the tetrahedron.

Thus, in whatever position the tetrahedron be originally placed, its centre of gravity will move in a vertical line, and make finite oscillations in that line.

The force acting downwards on the solid at any time

= the weight of the solid – weight of fluid displaced

$$= g\sigma V - g \iiint (c + \mu z) dx dy dz$$

(if V be the volume of the tetrahedron, σ its density)

$$= g\sigma V - gcV,$$

since the centre of gravity is the origin of coordinates, and therefore

$$\iiint z \, dx \, dy \, dz = 0.$$

Hence the equation of oscillating motion is

$$V\sigma \frac{d^2 z'}{dt^2} = g\sigma V - g\mu z' V,$$

since $c = \mu z'$;

$$\text{or } \frac{d^2 z'}{dt^2} + \frac{g\mu}{\sigma} z' = g,$$

and the time of an oscillation

$$= 2\pi \left(\frac{\sigma}{g\mu} \right)^{\frac{1}{2}}.$$

Now h , the depth of the centre of gravity in the position, is the value of z' in the above equation, when $\frac{d^2 z'}{dt^2} = 0$;

$$\therefore \frac{g\mu}{\sigma} h = g,$$

$$\text{or } h = \frac{\sigma}{\mu},$$

and the time of an oscillation

$$= 2\pi \left(\frac{h}{g} \right)^{\frac{1}{2}}.$$

This proposition is equally true of any homogeneous regular solid.

GEOMETRICAL OPTICS.

1848.

1. Compare the brightness of the Earth as seen from Venus with the brightness of Venus as seen from the Earth, supposing the sizes and reflecting powers of the two bodies equal.

Let S, E, V , (fig. 111) be the respective positions of the Sun, and the centres of the Earth and Venus, at the time when their brightness is to be compared.

From E draw the straight lines Ea, Eb perpendicular to ES and EV in the plane of the ecliptic, and similarly Vc, Vd perpendicular to VS and VE : the part of the Earth seen from Venus will be contained between planes perpendicular to the ecliptic through Ea, Eb ; and the part of Venus seen from the Earth between the planes Vc, Vd .

Let Q be the quantity of light that falls upon a unit of the surface of Venus which has the Sun in its zenith. To find the quantity of light reflected to the Earth from any element δS of the surface of Venus.

Let the latitude and longitude of the element δS , referred to the Sun as origin, and plane of the ecliptic as plane of longitude, be θ and ϕ . The quantity of light reflected to the Earth from δS will

$$\begin{aligned} &= Q\delta S \times \text{cosine z. D. of Sun} \times \text{cosine z. D. of Earth} \\ &= Q\delta S \cdot \cos\theta \cos\phi \cdot \cos\theta \cos(V-\phi), \quad V = \angle SVE, \\ &= Q \cdot r^2 \delta\theta \cos\theta \delta\phi \cdot \cos^2\theta \cos\phi \cos(V-\phi), \end{aligned}$$

if r = radius of the Earth or Venus.

Hence the whole light reflected to the Earth

$$\begin{aligned} &= Qr^2 \iint \cos^3\theta \cos\phi \cos(V-\phi) d\theta d\phi \\ &= \frac{1}{2} Qr^2 \iint \cos^3\theta \{\cos V + \cos(V-2\phi)\} d\theta d\phi \\ &= \frac{1}{2} Qr^2 \int \cos^3\theta \{\cos V \cdot \phi - \frac{1}{2} \sin(V-2\phi) + C\} d\theta: \end{aligned}$$

$$\begin{array}{l} \text{from } \phi = (V - \frac{1}{2}\pi) \\ \text{to } \phi = \frac{1}{2}\pi \end{array} \Bigg\}$$

$$\begin{aligned} &= \frac{1}{2} Q r^2 \int \cos^3 \theta \{ \cos V (\pi - V) + \frac{1}{2} \sin (\pi - V) \\ &\quad - \frac{1}{2} \sin (V - \pi) \} d\theta \\ &= \frac{1}{2} Q r^2 \{ (\pi - V) \cos V + \sin V \} \int \cos^3 \theta d\theta \\ &= \frac{1}{8} Q r^2 \{ (\pi - V) \cos V + \sin V \} \int (\cos 3\theta + 3 \cos \theta) d\theta : \end{aligned}$$

$$\begin{array}{l} \text{from } \theta = -\frac{1}{2}\pi \\ \text{to } \theta = +\frac{1}{2}\pi \end{array} \Bigg\}$$

$$= \frac{5}{6} Q r^2 \{ (\pi - V) \cos V + \sin V \}.$$

Similarly, the whole light reflected from the Earth to Venus

$$= \frac{5}{6} Q' r'^2 \{ (\pi - E) \cos E + \sin E \},$$

when Q' is the quantity of light that falls upon a unit of the Earth's surface which has the Sun in its zenith;

$$\therefore \frac{Q'}{Q} = \frac{SV^2}{SE^2},$$

and the required ratio

$$= \frac{SV^2}{SE^2} \cdot \frac{(\pi - E) \cos E + \sin E}{(\pi - V) \cos V + \sin V}.$$

2. Find the geometrical focus of a pencil of rays refracted through a hollow glass sphere, whose external and internal radii are r, r' respectively.

Let u, v_1, v_2, v_3 , and v , be the distances from the centre of the sphere of the foci before the 1st and after the 1st, 2nd, 3rd, and 4th refractions respectively.

Then, by the common formula,

$$\frac{1}{v_1} = -\frac{\mu - 1}{r} + \frac{\mu}{u} \dots\dots\dots (1),$$

$$\frac{1}{v_2} = -\frac{\frac{1}{\mu} - 1}{r'} + \frac{1}{\mu} \cdot \frac{1}{v_1},$$

$$\text{or } \frac{\mu}{v_2} = \frac{\mu - 1}{r'} + \frac{1}{v_1} \dots\dots\dots(2),$$

$$\frac{1}{v_3} = \frac{\mu - 1}{r'} + \frac{\mu}{v_2} \dots\dots\dots(3),$$

$$\text{and } \frac{1}{v_4} = \frac{\frac{1}{\mu} - 1}{r} + \frac{1}{\mu} \cdot \frac{1}{v_3},$$

$$\text{or } \frac{\mu}{v_4} = -\frac{\mu - 1}{r} + \frac{1}{v_3} \dots\dots\dots(4).$$

Adding (1), (2), (3), and (4),

$$\frac{\mu}{v_4} = 2(\mu - 1) \left(\frac{1}{r'} - \frac{1}{r} \right) + \frac{\mu}{u};$$

and for glass, $\mu = \frac{3}{2}$;

$$\therefore \frac{1}{v_4} = \frac{2}{3} \left(\frac{1}{r'} - \frac{1}{r} \right) + \frac{1}{u},$$

which determines v_4 .

3. Light, proceeding from a given point P , suffers any number of reflections and refractions: if consecutive rays of a given colour come out parallel, in the direction determined by angular coordinates θ, ϕ , shew that $\frac{d\theta}{d\mu}, \frac{d\phi}{d\mu}$ may be obtained by differentiating as if the differently coloured rays which severally come out parallel to their consecutives started from P in the same direction.

Application. In the case of the rainbow of the p^{th} order, given

$$D = p\pi + 2\phi - 2(p + 1)\phi', \quad \sin\phi = \mu \sin\phi',$$

find the order of the colours.

In general, if θ, ϕ be the coordinates upon emergence of the ray whose coordinates as it proceeded from P were θ', ϕ' , and if μ be the refractive index of the ray,

$$\theta = f_1(\theta', \phi', \mu),$$

$$\phi = f_2(\theta', \phi', \mu).$$

Now, if θ and ϕ be the coordinates of the rays of refractive index μ which come out parallel to their consecutives, θ' , ϕ' may be found in terms of μ from the equations

$$\frac{d\theta}{d\theta'} + \frac{d\theta}{d\phi'} \cdot \frac{d\phi'}{d\theta'} = 0,$$

$$\frac{d\phi}{d\theta'} + \frac{d\phi}{d\phi'} \cdot \frac{d\phi'}{d\theta'} = 0;$$

those consecutive rays being supposed to come out parallel which before incidence lie in a plane defined by the particular value of $\frac{d\phi'}{d\theta'}$.

Now, supposing θ and ϕ to retain the particular meaning assigned to them above, since θ' and ϕ' are now functions of μ ,

$$\frac{d\theta}{d\mu} = \left(\frac{d\theta}{d\theta'} + \frac{d\theta}{d\phi'} \cdot \frac{d\phi'}{d\theta'} \right) \frac{d\theta'}{d\mu} + \frac{d\theta}{d\mu},$$

$$\frac{d\phi}{d\mu} = \left(\frac{d\phi}{d\theta'} + \frac{d\phi}{d\phi'} \cdot \frac{d\phi'}{d\theta'} \right) \frac{d\theta'}{d\mu} + \frac{d\phi}{d\mu}.$$

The first terms in these expressions for $\frac{d\theta}{d\mu}$, $\frac{d\phi}{d\mu}$ correspond to the variation of the direction of emergence due to the variation of the direction of incidence; the second terms correspond to the variation of the direction of emergence due to the variation of μ in the differently coloured rays. But we have seen that the first terms are each equal to zero; hence the whole variation of θ and ϕ is due to the variation of μ in the differently coloured rays, or we may obtain $\frac{d\theta}{d\mu}$, $\frac{d\phi}{d\mu}$ by differentiating as if the differently coloured rays which severally come out parallel to their consecutives started from P in the same direction.

In the application to the case of the rainbow all the incident rays are parallel; we may, however, differentiate for $\frac{d\theta}{d\mu}$, $\frac{d\phi}{d\mu}$ as if all the angles of incidence of rays which come out parallel to their consecutives were equal.

Here each ray which comes to the eye moves in the same plane; and since the rays incident upon the raindrop are all parallel, the angular coordinate after emergence will be D , the deviation. Hence, to find $\frac{dD}{d\mu}$ we differentiate D , considering ϕ constant;

$$\therefore \frac{dD}{d\mu} = -2(p+1) \frac{d\phi'}{d\mu},$$

$$\text{and } \sin \phi = \mu \sin \phi';$$

$$\therefore 0 = \sin \phi' + \mu \cos \phi' \frac{d\phi'}{d\mu},$$

$$\text{and } \frac{dD}{d\mu} = \frac{2(p+1)}{\mu} \tan \phi' \text{ is positive:}$$

hence the red rays which come out parallel to their consecutives will be more deviated than the violet rays which come out parallel to their consecutives.

It only remains to find in which direction the rays which form the rainbow have been deviated. To ascertain this, we must differentiate D with respect to ϕ , considering μ constant, and put

$$\frac{dD}{d\phi} = 0.$$

This equation will give us a value of ϕ , which substituted in D will determine the amount of deviation of the rays of the refractive index μ , by which the corresponding part of the rainbow is seen: let this value be

$$D = 2m\pi + \psi:$$

then, if ψ be $< \pi$, the deviation at the first refraction will be towards the eye, and the red rays will appear on the inside of the arch: if ψ be $> \pi$, the deviation at the first refraction will be from the eye, and the red rays will appear on the outside of the arch.

4. Every diameter (d) of the extreme boundary of a spherical reflector subtends a right angle at C the centre of the sphere: supposing parallel rays (inclined at an angle α to the

axis of the reflector) to be incident upon every point of the extreme boundary, shew that the section of the reflected pencil made by a plane passing through C and perpendicular to the axis, will be an hyperbola, whose axes are d and $d \cot \alpha$.

Let CO (fig. 112) be the axis of the mirror AOB ; let Cx , parallel to the direction of incident rays, be taken for axis of x ; Cy , perpendicular to it in the plane OAx , for axis of y , and Cz perpendicular to Cx , Cy for axis of z .

Let Cx pierce the mirror at the point o : join P , any point in the boundary of the mirror by arcs of great circles, with O , o : call Po , PoA , θ and ϕ : OP will be 45° .

The ray reflected from P will pass through the point D of Co , such that the perpendicular from D upon CP bisects CP , i.e. through the point $(\frac{1}{2}a \sec \theta, 0, o)$: also the coordinates of P are $a \cos \theta$, $a \sin \theta \cos \phi$, $a \sin \theta \sin \phi$. Hence the equation of the reflected ray is

$$\frac{x - \frac{1}{2}a \sec \theta}{a \cos \theta - \frac{1}{2}a \sec \theta} = \frac{y}{a \sin \theta \cos \phi} = \frac{z}{a \sin \theta \sin \phi} = \lambda \text{ suppose... (1).}$$

Also, from the triangle OPo ,

$$\cos 45^\circ = \cos \theta \cos \alpha - \sin \theta \sin \alpha \cos \phi \dots\dots\dots (2).$$

Our object will be to eliminate θ , ϕ from these equations, and find the relation between η and z when we have written

$$x = \eta \sin \alpha,$$

$$y = \eta \cos \alpha:$$

the equation so formed will evidently be the equation of the curve in which the plane through C cuts the surface formed by the refracted rays.

From (1),

$$x \cos \theta - \frac{1}{2}a = a\lambda \cos^2 \theta - \frac{1}{2}a\lambda;$$

$$\therefore a^2 \lambda^2 \cos^2 \theta - a\lambda x \cos \theta + \frac{1}{4}x^2 = \frac{1}{2}a^2 \lambda (\lambda - 1) + \frac{1}{4}x^2;$$

$$\therefore a\lambda \cos \theta = \frac{1}{2}x + \left\{ \frac{1}{2}a^2 \lambda (\lambda - 1) + \frac{1}{4}x^2 \right\}^{\frac{1}{2}} \dots\dots\dots (3).$$

Hence equation (2) becomes, multiplying by $a\lambda$,

$$\frac{1}{2^{\frac{1}{2}}} a\lambda = \left\{ \frac{1}{2}x + \left(\frac{1}{2}a^2 \lambda (\lambda - 1) + \frac{1}{4}x^2 \right)^{\frac{1}{2}} \right\} \cos \alpha - y \sin \alpha,$$

$$\text{or } \left\{ \frac{1}{2}a^2\lambda(\lambda-1) + \frac{1}{4}x^2 \right\}^{\frac{1}{2}} \cos\alpha = \frac{1}{2^{\frac{1}{2}}} a\lambda + y \sin\alpha - \frac{1}{2}x \cos\alpha,$$

$$\text{or } \left\{ \frac{1}{2}a^2\lambda(\lambda-1) + \frac{1}{4}\eta^2 \sin^2\alpha \right\}^{\frac{1}{2}} \cos\alpha = \frac{1}{2^{\frac{1}{2}}} a\lambda + \frac{1}{2}\eta \sin\alpha \cos\alpha \dots (4):$$

hence, squaring,

$$\frac{1}{2}a^2\lambda(\lambda-1) \cos^2\alpha = \frac{1}{2}a^2\lambda^2 + \frac{1}{2^{\frac{1}{2}}} a\lambda \eta \sin\alpha \cos\alpha;$$

$$\therefore a\lambda \sin^2\alpha + a \cos^2\alpha = -2^{\frac{1}{2}}\eta \sin\alpha \cos\alpha,$$

$$\text{and } a\lambda = -2^{\frac{1}{2}}\eta \cot\alpha - a \cot^2\alpha.$$

Again, from equations (1),

$$a^2\lambda^2 \sin^2\theta = y^2 + z^2:$$

adding this equation to (3)²,

$$a^2\lambda^2 = \frac{1}{2}x^2 + y^2 + z^2 + \frac{1}{2}a^2\lambda(\lambda-1) + \left\{ \frac{1}{2}a^2\lambda(\lambda-1) + \frac{1}{4}x^2 \right\}^{\frac{1}{2}} x,$$

or by (4),

$$\frac{1}{2}a^2\lambda(\lambda+1) = \frac{1}{2}\eta^2 + \frac{1}{2}\eta^2 \cos^2\alpha + z^2 + \frac{1}{\cos\alpha} \left(\frac{1}{2^{\frac{1}{2}}} a\lambda + \frac{1}{2}\eta \sin\alpha \cos\alpha \right) \eta \sin\alpha;$$

or retaining only the second powers of η and z ,

$$\eta^2 \cot^2\alpha = \frac{1}{2}\eta^2(1 + \cos^2\alpha) + z^2 - \eta^2 + \frac{1}{2}\eta^2 \sin^2\alpha + \&c.,$$

or $\eta^2 \cot^2\alpha - z^2 = \&c.;$

shewing that the required locus is a hyperbola, the ratio of whose axes is $\cot\alpha$.

Now the axis in the plane OCx is evidently d : hence the axes are d and $d \cot\alpha$.

1849.

1. A ray of light is incident upon one of two reflectors inclined to each other at an angle $\frac{\pi}{n}$, in a direction parallel to a line which is at right angles to their intersection, and bisects the angle between them: supposing the intensity of a ray reflected at an angle ϕ to be to that of the incident ray as $e \cos\phi$ to 1, shew that the intensity of the ray after it has suffered n reflexions will be to that of the incident ray as e^n to 2^{n-1} .

From the problem on p. 60 it appears that, if n be even, the successive angles of reflexion are complementary of

$$\frac{1}{2} \cdot \frac{\pi}{n}, \quad \frac{3}{2} \cdot \frac{\pi}{n} \dots \frac{n-1}{2} \cdot \frac{\pi}{n}, \quad \frac{n-1}{2} \cdot \frac{\pi}{n} \dots \frac{3}{2} \cdot \frac{\pi}{n}, \quad \frac{1}{2} \cdot \frac{\pi}{n}.$$

Hence the intensity of the ray after n reflexions : that of the incident ray

$$:: e^n \left(\sin \frac{1}{2} \frac{\pi}{n} \cdot \sin \frac{3}{2} \frac{\pi}{n} \dots \sin \frac{n-1}{2} \cdot \frac{\pi}{n} \right)^2 : 1.$$

Similarly, when n is odd, this ratio is

$$e^n \left(\sin \frac{1}{2} \frac{\pi}{n} \cdot \sin \frac{3}{2} \frac{\pi}{n} \dots \sin \frac{n}{2} \cdot \frac{\pi}{n} \right)^2 : 1.$$

Both these ratios may be expressed by the general formula

$$e^n \sin \frac{1}{2} \frac{\pi}{n} \cdot \sin \frac{3}{2} \frac{\pi}{n} \dots \sin \frac{2n-1}{2} \frac{\pi}{n} : 1;$$

and in Hymers' *Theory of Equations*, Art. 22, Ex. 20, it appears, by making $\theta = 0$, that

$$\sin \frac{1}{2} \frac{\pi}{n} \sin \frac{3}{2} \frac{\pi}{n} \dots \sin \frac{2n-1}{2} \frac{\pi}{n} = \frac{1}{2^{n-1}},$$

whether n be odd or even : hence each of the above ratios

$$= e^n : 2^{n-1}.$$

2. A transparent medium is bounded by two parallel planes; the refractive index is constant throughout any plane parallel to the bounding planes, but varies continuously in the direction of the normal to those planes: shew how to find the path of a ray of light through such a medium, and prove that in passing through a section for which the refractive index is a maximum or a minimum, the path will in general have a point of contrary flexure.

It is evident that the path of any ray will lie in one plane: in this plane take two lines, one perpendicular to the bounding planes, the other parallel to them as axes of x and y respectively.

At the point (xy) let the inclination of the path to the axis of x be ϕ , and let the refractive index be μ : at an adjacent point $(x + \delta x, y + \delta y)$, let these be $\phi + \delta\phi$, $\mu + \delta\mu$: then, by the law of refraction,

$$\begin{aligned}\sin\phi &= \frac{\mu + \delta\mu}{\mu} \sin(\phi + \delta\phi) \\ &= \left(1 + \frac{\delta\mu}{\mu}\right) (\sin\phi + \cos\phi\delta\phi), \\ \text{or } 0 &= \cos\phi\delta\phi + \frac{1}{\mu} \sin\phi\delta\mu;\end{aligned}$$

therefore, proceeding to the limit,

$$\begin{aligned}\frac{d\phi}{d\mu} + \frac{1}{\mu} \tan\phi &= 0, \\ \text{or } \cot\phi \frac{d\phi}{dx} + \frac{1}{\mu} \frac{d\mu}{dx} &= 0;\end{aligned}$$

$$\therefore \log \sin\phi + \log\mu = \text{constant} = \log C,$$

$$\therefore \sin\phi = \frac{C}{\mu},$$

$$\text{or } 1 + \left(\frac{dx}{dy}\right)^2 = \left(\frac{\mu}{C}\right)^2,$$

$$\text{and } \frac{dy}{dx} = \left\{\left(\frac{\mu}{C}\right)^2 - 1\right\}^{-\frac{1}{2}};$$

which, since μ is a known function of x , is a differential equation for the determination of the path.

When μ is a maximum or minimum, we have $\frac{d\mu}{dx} = 0$,

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} &= -\frac{\mu}{C^2} \left\{\left(\frac{\mu}{C}\right) - 1\right\}^{-\frac{3}{2}} \frac{d\mu}{dx} \\ &= 0,\end{aligned}$$

and as the ray passes through such a section, its path usually suffers inflexion.

1850.

1. If a string be wrapped round a glass prism, whose section is an equilateral triangle, so as to be always inclined at the same

angle to the axis of the prism, the portion of the string seen by internal reflexion will appear to be parallel to the portions seen directly.

Let AB (fig. 113) be a portion of the string seen directly, BC a portion seen by internal reflexion at the surface AC of the prism: and first, let the eye be in such a position that the small pencil of rays by which any point of BC very near B is seen, shall pass through some point of the surface ABB' very near to B' , a point in the edge AB' . Then, if the eye be at a considerable distance from the prism so that the axes of small visual pencils may be considered parallel to each other, any point of BC very near C will be seen by a small pencil which passes through the surface ABB' at some point very near C' a point in the edge BC' . Let $B'E$, $C'E$ be the directions of these small pencils upon emergence. Now the plane $CC'E$ is parallel to the plane $BB'E$; and if we draw a plane through A parallel to these, the plane $BB'E$ will be equidistant from the other two, since the prism is equilateral, and AB , BC equally inclined to its axis; hence $BC' = AB'$, and $B'C'$ is parallel to AB : hence BC appears parallel to AB .

Now let the eye be moved in any manner without approaching too near the prism; it may easily be seen that the locus of the points where the rays from BC to the eye cross the plane ABB' is parallel to $B'C'$, and therefore to AB . Hence BC will always appear parallel to AB ; and the same may be shewn of any other portion of the string seen by internal reflexion.

2. A rectangular box, at the bottom of which is a plane mirror, contains an unknown quantity of water; from the angle at which a ray of light must enter through one of two small holes in the lid in order that after refraction and reflexion it may emerge at the other, determine the height of the water in the box.

Let a be the height of the box, x the depth of the water, $2b$ the distance between the small holes in the lid; ϕ , ϕ' the angles of incidence and refraction when the ray enters the water,

they will also be the angle of refraction and incidence as it emerges from it: hence we have

$$\begin{aligned} b &= (a - x) \tan \phi + x \tan \phi', \\ \sin \phi &= \mu \sin \phi'; \\ \therefore x &= \frac{a \tan \phi - b}{\tan \phi - \tan \phi'} \\ &= \frac{a \tan \phi - b}{\tan \phi - \frac{\sin \phi}{(\mu^2 - \sin^2 \phi)^{\frac{1}{2}}}}, \end{aligned}$$

whence x is known from ϕ , which is the angle the ray makes with the vertical upon entering the hole in the lid.

3. If a luminous point be reflected by a small plane mirror, so as to be seen by an eye in a given position, and the mirror move in such a way that the luminous point always appears to be upon a given conical surface, of which the point is the vertex and a line through the eye the axis; find the form of the surface upon which the small mirror must always be situated.

Let O, E (fig. 114) be the position of the luminous point and eye respectively, M any position of the mirror, P the corresponding position of the image of O : then will EMP be a straight line, and $MP = MO$. Take OEx for axis of x , Oy perpendicular to it for that of y ; x, y the coordinates of M ; x', y' those of P . Then

$$\begin{aligned} MP &= MO; \\ \therefore (x' - x)^2 + (y' - y)^2 &= x^2 + y^2, \\ \text{or } x'^2 + y'^2 - 2xx' - 2yy' &= 0 \dots \dots \dots (1): \end{aligned}$$

E, M, P , are in the same straight line,

$$\therefore \frac{y}{x - a} = \frac{y'}{x' - a} \dots \dots \dots (2).$$

Let the equation to the generating line of the cone on which P is situated, be

$$y' = mx' \dots \dots \dots (3):$$

between (1), (2), (3), we have to eliminate x', y' .

From (2) and (3),

$$\frac{x' - a}{x'} = \frac{m(x - a)}{y};$$

$$\therefore x' = \frac{ay}{y - m(x - a)}.$$

From (1) and (3),

$$(1 + m^2)x' - 2(x + my) = 0,$$

$$\text{or } 2(x + my)\{y - m(x - a)\} - (1 + m^2)ay = 0;$$

$$\therefore 2m(y^2 - x^2) + 2(1 - m^2)xy + 2max - (1 - m^2)ay = 0,$$

the equation to an equilateral hyperbola; and the required surface is an equilateral hyperboloid of revolution.

4. If the earth, supposed spherical, were covered to a depth h with water, h being small compared with the earth's radius, shew that the height to which a person must be raised above the surface of the water in order to see as far below the horizon as when he was on the surface of the earth is $\frac{h^2}{2r(\mu^2 - 1)}$ nearly, μ being the index of refraction for water.

Let O (fig. 115) be the centre of the earth, A the station of the observer on the earth's surface: in order to see as far below the horizon as possible he must look in the direction AB , such that OBA is the critical angle; he will then see objects situated in the line BC , if BC be a tangent to the surface of the water at B . Hence P , the raised position of the observer, must be the intersection of CB , OA . Let $AOB = \theta$: then, if $AP = x$,

$$\frac{r + h}{r + x} = \cos \theta,$$

or, since θ , $\frac{h}{r}$, $\frac{x}{r}$, are small,

$$1 - \frac{x - h}{r} = 1 - \frac{1}{2}\theta^2;$$

$$\therefore \frac{x - h}{r} = \frac{1}{2}\theta^2.$$

Also from the triangle OAB ,

$$\begin{aligned}\frac{r+h}{r} &= \frac{\sin\left(\sin^{-1}\frac{1}{\mu} + \theta\right)}{\sin\sin^{-1}\frac{1}{\mu}} \\ &= \cos\theta + (\mu^2 - 1)^{\frac{1}{2}} \sin\theta, \\ \text{or } 1 + \frac{h}{r} &= 1 + (\mu^2 - 1)^{\frac{1}{2}} \theta, \\ \text{or } \theta &= \frac{h}{(\mu^2 - 1)^{\frac{1}{2}} r}; \\ \therefore \frac{x-h}{r} &= \frac{1}{2}\theta^2 = \frac{h^2}{2r^2(\mu^2 - 1)}, \\ \text{or } x-h &= \frac{h^2}{2r(\mu^2 - 1)},\end{aligned}$$

the required distance to which the man must be raised above the surface of the water.

1851.

A number of vertical plane reflectors are placed together so as to meet a horizontal plane in a polygon of n sides: find the path of a ray of light which, after reflexion at the n plane reflectors in succession, will continue to proceed in its original course.

Shew also that when there are four reflectors the problem is either indeterminate or impossible; and that when the number of reflectors is even, and the polygon capable of being inscribed in a circle, the problem is indeterminate.

Let $\theta_1, \theta_2 \dots \theta_n$ be the complements of the successive angles of incidence or reflexion, $\alpha_1, \alpha_2 \dots \alpha_n$ the angles of the polygon: then $\theta_1, \theta_2, \alpha_1$ are the angles of a triangle;

$$\left. \begin{aligned}\therefore \theta_1 + \theta_2 &= \pi - \alpha_1: \\ \text{so } \theta_2 + \theta_3 &= \pi - \alpha_2 \\ \dots\dots\dots &= \dots\dots\dots \\ \theta_n + \theta_1 &= \pi - \alpha_n\end{aligned} \right\} \dots\dots\dots (A).$$

Again, let $u_1 = 0, u_2 = 0, \dots u_n = 0$, be the equations of the successive parts of the ray's path, $a_1 = 0, a_2 = 0, \dots a_n = 0$, the equations of the sides of the polygon; the equations being all in such a form that u considered as a function of x and y is the distance of the point (xy) from the line $u = 0$.

Now a_1 is the external bisector of u_1, u_2 , whence

$$u_1 + u_2 = 2a_1 \cos \theta_1.$$

For let Ou_1, Ou_2, Oa_1 , (fig. 116) be the lines u_1, u_2, a_1 ; take any point P , join OP , and draw Pr, Ps, Pt , perpendicular to these lines respectively. Then

$$\begin{aligned} Pr + Pt &= OP(\sin PO_r + \sin PO_t) \\ &= 2OP \sin \frac{1}{2}(PO_r + PO_t) \cos \frac{1}{2}(PO_r - PO_t) \\ &= 2OP \sin PO_s \cos tOs \\ &= 2Ps \cos \theta_1. \end{aligned}$$

Hence, if x, y , the coordinates of P , be substituted in u_1, u_2, a_1 , we shall have

$$u_1 + u_2 = 2a_1 \cos \theta_1,$$

and the same may be shewn wherever the point P is taken; hence generally,

$$\left. \begin{aligned} u_1 + u_2 &= 2a_1 \cos \theta_1; \\ \text{so } u_2 + u_3 &= 2a_2 \cos \theta_2 \\ \dots\dots\dots &= \dots\dots\dots \\ u_n + u_1 &= 2a_n \cos \theta_n \end{aligned} \right\} \dots\dots\dots (B).$$

From equations (A) $\theta_1, \theta_2, \dots \theta_n$ must be found, and thence $u_1, u_2, \dots u_n$ from equations (B), and the path of the ray will then be fully determined.

If there be four reflectors we find, by adding the 1st and 3rd of equations (A),

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi - \alpha_1 - \alpha_3;$$

similarly, by adding the 2nd and 4th,

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi - \alpha_2 - \alpha_4;$$

hence we must have

$$\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = \pi,$$

or the quadrilateral must be inscribable in a circle in order that the problem may be possible.

If however this condition is satisfied, still the problem is indeterminate; for if we treat equations (B) as above, we find

$$a_1 \cos \theta_1 + a_3 \cos \theta_3 = a_2 \cos \theta_2 + a_4 \cos \theta_4 \dots\dots\dots (1).$$

This is an identical equation; it will therefore lead us, by means of equating the coefficients of x and y on the two sides of the equation, to three conditions between $\theta_1, \theta_2, \theta_3, \theta_4$, and constants: between these constants there are also two relations arising from the circumstance that the quadrilateral may be inscribed in a circle, and the three conditions between $\theta_1, \theta_2, \theta_3, \theta_4$, and constants amount to only one equation independent of equations (A).

This condition, together with equations (A), will determine $\theta_1, \theta_2, \theta_3, \theta_4$. But since we have derived the condition (1) from equation (B), it shews that those equations are equivalent to only three independent equations, and u_1, u_2, u_3, u_4 , are therefore indeterminate. The direction only of the different parts of the path of the ray can be determined; with these directions any position will satisfy the problem.

Similarly, if there be any even number of reflectors, we may, from equations (A), deduce the condition

$$\alpha_1 + \alpha_3 + \dots + \alpha_{n-1} = \alpha_2 + \alpha_4 + \dots + \alpha_n,$$

or the sums of the alternate angles must be equal: this condition is satisfied if the polygon can be inscribed in a circle, and the problem is then possible.

The problem is still indeterminate, for from equations (B) we may deduce the condition

$$a_1 \cos \theta_1 + a_3 \cos \theta_3 + \dots + a_{n-1} \cos \theta_{n-1} = a_2 \cos \theta_2 + a_4 \cos \theta_4 + \dots + a_n \cos \theta_n :$$

this identical equation will lead, as before, to one equation of condition independent of equations (A) between $\theta_1, \theta_2, \dots \theta_n$ and constants, which, with equations (A), serves to determine

$\theta_1, \theta_2, \dots \theta_n$. Still equations (B) are equivalent to only $n - 1$ independent equations, and $u_1, u_2, \dots u_n$ are therefore indeterminable. In fact, we have shewn in Problem 2, page 59, that 'if a ray of light, after being reflected any number of times in one plane, at any number of plane surfaces, return on its former course, the same will be true of any ray parallel to the former which is reflected at the same surfaces in the same order, provided the number of reflexions be even.'

ASTRONOMY.

1848.

1. If there had been no stars, how might the absolute periodic times of the Earth and planets have been determined, even if the equator had coincided with the ecliptic?

We might first have determined the synodic time (T) of the Earth and any superior planet by observing the interval between successive conjunctions. Let E , P be the periodic times of the Earth and the planet,

$$\therefore \frac{2\pi}{E} - \frac{2\pi}{P} = \text{relative angular velocity of} \\ \text{the Earth and planet}$$

$$= \frac{2\pi}{T},$$

$$\text{or } E = T \left(1 - \frac{E}{P} \right).$$

We might then have observed the elongation from the Sun of P and the other planets at their points of station: this gives the ratio of the distances of the Earth and each planet from the Sun, and therefore, by Kepler's law that the squares of the periods are as the cubes of the mean distances, it would give approximately the ratio of the periods or $\frac{E}{P}$ for each of the planets; whence from above E , and the periods of all the planets, would be known.

2. A star map is laid down on the gnomonic projection, the plane of projection being parallel to the equator: give a graphical solution of the problem, to determine the time at a

known place by observing when two stars laid down in the map are in the same vertical plane.

Since the place is known, we may draw about the centre of the map the circle described by the projection of the place on account of the Earth's daily rotation. Let a line through the two stars intersect this circle in two points A and B . Then it is plain from the method of projection, that when the projection of the place is at A or B , the two stars are in the same vertical.

Let the differences of longitude of the Sun in its place among the stars on the day of observation, and the points A and B , be observed; this longitude, converted into hours at the rate of 15 degrees to an hour, will give the interval since last noon or the true solar time.

3. Shew that at the equinoxes the extremity of the shadow of the style of a vertical south dial will trace upon the dial-plate a horizontal straight line at a distance $a \operatorname{cosec} l$ from the upper extremity of the style, a being the length of the style and l the latitude of the place.

On the day of the equinox the Sun appears to move in a great circle of the celestial sphere. We may consider the extremity of the style as the centre of that sphere, or that the Sun moves in a plane through the extremity. The normal to this plane lies in the vertical plane through the north and south points, therefore its intersection with the dial-plate will be a horizontal straight line; this is the line traced out by the extremity of the shadow of the style.

Let the plane of the paper be the vertical plane containing the style AB (fig. 117). Draw AC vertical and BC perpendicular to AB : then, since AB is parallel to the Earth's axis, and the Sun is in the equator, C is the extremity of the shadow at noon, and AC is the distance of the horizontal line from A : it $= AB \operatorname{cosec} ACB = a \operatorname{cosec} l$.

1850.

1. If a rod be fixed into a vertical wall which faces the south and the shadow of it be cast upon the wall by the Sun, find the curve upon which the shadow of the end of the rod will be situated every day at mean noon, the Sun being supposed to move uniformly in the ecliptic with his mean motion.

The mean Sun is situated on the equator at the same distance from \mathfrak{r} as the true Sun; it is mean noon when this mean Sun is due south.

Let S, S' (fig. 118) be the true and mean Suns, then $\mathfrak{r} S = \mathfrak{r} S'$, and if we draw SD an arc of a great circle perpendicular to the equator, and call $\mathfrak{r} S, \mathfrak{r} D, SD, L, a, \delta$, respectively, we have

$$S'D = \mathfrak{r} S - \mathfrak{r} D = L - a.$$

Now let E be the extremity of the rod, O its shadow when the Sun is in \mathfrak{r} , P the position of its shadow when the Sun is at S , and S' on the meridian, *i.e.* due south. Then if we draw ON horizontal, NP vertical in the plane of the wall, and join EO, EN, EP ; $OEN = S'D, NEP = SD, EPN = l - \delta$, where l = latitude of the place. Call On, x, NP, y, EO, d ,

$$\therefore x = d \tan S'd,$$

$$y = \frac{(d^2 + x^2)^{\frac{1}{2}} \sin SD}{\sin(l - \delta)};$$

$$\text{or } x = d \tan(L - a) = d \frac{\tan L - \tan a}{1 + \tan a \tan L}$$

$$= d \frac{\tan a - \cos \omega \tan a}{\cos \omega + \tan^2 a}, \text{ since } \cos \omega \tan L = \tan a$$

$$= d \frac{(1 - \cos \omega) \tan a}{\cos \omega + \tan^2 a} \dots \dots \dots (1);$$

$$\text{and } y = (d^2 + x^2)^{\frac{1}{2}} \frac{\sin \delta}{\sin(l - \delta)} = (d^2 + x^2)^{\frac{1}{2}} \frac{1}{\sin l \cot \delta - \cos l}$$

$$= (d^2 + x^2)^{\frac{1}{2}} \frac{\sin a}{\sin l \cot \omega - \cos l \sin a}, \text{ since } \sin a \cot \delta = \cot \omega \dots (2).$$

We have now to eliminate a between (1) and (2).

From (2)

$$\frac{\sin l \cot \omega}{\sin \alpha} - \cos l = \frac{(d^2 + x^2)^{\frac{1}{2}}}{y},$$

$$\text{or } \sin \alpha = \sin l \cot \omega \frac{y}{(d^2 + x^2)^{\frac{1}{2}} + y \cos l}.$$

From (1)

$$(\cos \omega \cos^2 \alpha + \sin^2 \alpha) x = d(1 - \cos \omega) \sin \alpha \cos \alpha,$$

$$\text{or } \{\cos \omega + (1 - \cos \omega) \sin^2 \alpha\} x^2 = d^2 (1 - \cos \omega)^2 \sin^2 \alpha (1 - \sin^2 \alpha);$$

$$\begin{aligned} \therefore [\cos \omega \{(d^2 + x^2)^{\frac{1}{2}} + y \cos l\}^2 + (1 - \cos \omega) \sin^2 l \cot^2 \omega \cdot y^2] x^2 \\ = (1 - \cos \omega)^2 \cot^2 \omega \sin^2 l d^2 y^2 [\{(d^2 + x^2)^{\frac{1}{2}} + y \cos l\}^2 - \sin^2 l \cot^2 \omega \cdot y^2], \end{aligned}$$

the equation to the required curve.

2. Suppose that during the day of the equinox, a man walks in a horizontal plane towards the Sun at a uniform rate; prove that the equation of the path described by him is

$$\sin \left(\frac{ny}{a \sec l} + l \right) = \frac{\sin l}{2} \left(\varepsilon \frac{nx}{a \tan l} + \varepsilon^{-1} \frac{nx}{a \tan l} \right),$$

where x and y are the coordinates of his position at any time, measured along and at right angles to his meridian at noon; l is his latitude, and a is the space he walks over while the Earth revolves through an angle n .

Deduce the particular cases of his being at the pole and at the equator.

Let ω be the angular velocity of rotation of the Earth about its axis; the angle apparently described by the Sun in the time t will be ωt , since the Sun is in the equator, it being the day of the equinox.

If α be the Sun's azimuth at time t ,

$$\frac{dy}{dx} = \tan \alpha.$$

Now l and ωt are the sides of a right-angled triangle, supposing the man to start at noon, of which the angle opposite ωt is α ;

$$\therefore \tan \alpha = \frac{\tan \omega t}{\sin l},$$

$$\therefore \frac{dy}{dx} = \frac{\tan \omega t}{\sin l};$$

$$\text{and } \frac{dy}{ds} = \frac{\tan \omega t}{(\sin^2 l + \tan^2 \omega t)^{\frac{1}{2}}},$$

$$\frac{dx}{ds} = \frac{\sin l}{(\sin^2 l + \tan^2 \omega t)^{\frac{1}{2}}}.$$

$$\text{Also, } s = \frac{a}{n} \omega t,$$

$$\begin{aligned} \frac{dy}{ds} &= \frac{\tan \frac{n}{a} s}{\left(\sin^2 l + \tan^2 \frac{n}{a} s\right)^{\frac{1}{2}}} = \frac{\sin \frac{n}{a} s}{\left(\sin^2 l \cos^2 \frac{n}{a} s + 1 - \cos^2 \frac{n}{a} s\right)^{\frac{1}{2}}} \\ &= \frac{\sin \frac{n}{a} s}{\left(1 - \cos^2 l \cos^2 \frac{n}{a} s\right)^{\frac{1}{2}}} = \sec l \frac{\sin \frac{n}{a} s}{\left(\sec^2 l - \cos^2 \frac{n}{a} s\right)^{\frac{1}{2}}}; \end{aligned}$$

$$\therefore y = \frac{a}{n} \sec l \left\{ \cos^{-1} \left(\cos \frac{n}{a} s \cos l \right) - l \right\}; \quad \therefore y = 0 \text{ when } s = 0,$$

$$\text{or } \cos \frac{n}{a} s = \sec l \cos \left(\frac{ny}{a \sec l} + l \right) \dots \dots \dots (1).$$

$$\begin{aligned} \text{Again, } \frac{dx}{ds} &= \frac{\sin l \cos \frac{n}{a} s}{\left\{ \sin^2 l \left(1 - \sin^2 \frac{n}{a} s \right) + \sin^2 \frac{n}{a} s \right\}^{\frac{1}{2}}} \\ &= \frac{\sin l \cos \frac{n}{a} s}{\left(\sin^2 l + \cos^2 l \sin^2 \frac{n}{a} s \right)^{\frac{1}{2}}} \\ &= \tan l \frac{\cos \frac{n}{a} s}{\left(\tan^2 l + \sin^2 \frac{n}{a} s \right)^{\frac{1}{2}}} \end{aligned}$$

$$\therefore x = \frac{a}{n} \tan l \log \frac{\sin \frac{n}{a} s + \left(\tan^2 l + \sin^2 \frac{n}{a} s \right)^{\frac{1}{2}}}{\tan l}; \because x = 0 \text{ when } s = 0;$$

$$\therefore \sin \frac{n}{a} s + \left(\tan^2 l + \sin^2 \frac{n}{a} s \right)^{\frac{1}{2}} = \tan l \epsilon^{\frac{nx}{a \tan l}},$$

$$\therefore \sin \frac{n}{a} s - \left(\tan^2 l + \sin^2 \frac{n}{a} s \right)^{\frac{1}{2}} = -\tan l \epsilon^{-\frac{nx}{a \tan l}},$$

$$\therefore \left(\tan^2 l + \sin^2 \frac{n}{a} s \right)^{\frac{1}{2}} = \frac{\tan l}{2} \left(\epsilon^{\frac{nx}{a \tan l}} + \epsilon^{-\frac{nx}{a \tan l}} \right),$$

$$\text{or } \left(\sec^2 l - \cos^2 \frac{n}{a} s \right)^{\frac{1}{2}} = \frac{\tan l}{2} \left(\epsilon^{\frac{nx}{a \tan l}} + \epsilon^{-\frac{nx}{a \tan l}} \right),$$

$$\therefore \text{by (1) } \sin \left(\frac{ny}{a \sec l} + l \right) = \frac{\sin l}{2} \left(\epsilon^{\frac{nx}{a \tan l}} + \epsilon^{-\frac{nx}{a \tan l}} \right),$$

the required equation to the path.

We have in this solution considered l constant; if, however, the man be at the pole, l will $= \frac{1}{2}\pi$, and $\sec l$, $\tan l$ will be susceptible of great changes when l alters but very little: hence we must consider his motion as indefinitely small compared with that of the Sun, or $\frac{a}{n}$ indefinitely small: hence the above equation leads us to $x = 0$, $y = 0$; and the man merely stands at the pole looking towards the Sun.

If he be on the equator, $\tan l = 0$, and therefore $x = 0$, or he walks along the equator.

1851.

The declination of the Sun at two observations δ , δ' , and the Sun's motion in right ascension and longitude in the interval between the observations, are equal: shew that if ω be the obliquity, and α , l the Sun's right ascension and longitude at the first observation, $\cos \omega = \cos \delta \cos \delta'$; $\tan \alpha = \sin \delta \cot \delta'$; $\cot l = \sin \delta' \cot \delta$.

Let P , K (fig. 119) be the poles of the equator and ecliptic; S , S' the two positions of the Sun: then, since the differences

of the Sun's longitude and right ascension at S and S' are equal, $SS' = SPS'$. Let the angle $PSK = \phi$; draw KR , an arc of a great circle, to meet SP produced at right angles: then, in the triangle SPS' ,

$$\sin SS' \sin PSS' = \sin S'P \sin SPS',$$

$$\text{or } \cos \phi = \cos \delta'; \therefore \phi = \delta',$$

$$\text{and } KR = \phi = \delta':$$

$$\text{also } PR = \delta, \quad KP = \omega,$$

$$\angle KPR = 90 - \alpha, \quad \angle PKR = l.$$

Hence, by Napier's rules,

$$\cos \omega = \cos \delta \cos \delta' \dots\dots\dots (1),$$

$$\sin \delta = \tan \alpha \tan \delta',$$

$$\text{or } \tan \alpha = \sin \delta \cot \delta' \dots\dots\dots (2),$$

$$\sin \delta' = \cot l \tan \delta,$$

$$\text{or } \cot l = \sin \delta' \cot \delta \dots\dots\dots (3),$$

and (1), (2), (3), are the formulæ required to be proved.

DISTURBED MOTION.

1848.

Two bodies, P, P' (fig. 120), describe round a central body S circular orbits lying in one plane, the orbit of P being within that of P' ; prove that the disturbing force of P' on P , when wholly central and additious, will be equal to the disturbing force when P, P' are on opposite sides of S , provided SP' be a mean proportional between SP and $SP + SP'$.

Let P_1 be the position of P when the disturbing force (F_1) on P_1 is wholly central and additious;

$$\therefore F_1 = \frac{\mu}{P_1 P'^2} \cdot \frac{SP_1}{P_1 P'},$$

(where μ is the absolute force of attraction of P').

Under the condition that the force of attraction of P' on S and P_1 , perpendicular to SP_1 , is the same, *i.e.* that $SP'P_1$ is an equilateral triangle,

$$\therefore P'P_1 = P'S,$$

$$\text{or } F_1 = \frac{\mu \cdot SP_1}{SP'^3}.$$

Let F_2 be the disturbing force when P is in opposition to P' at P_2 ,

$$\therefore F_2 = \frac{\mu}{SP'^2} - \frac{\mu}{P_2 P'^2};$$

$$\text{and } F_2 = F_1$$

$$\text{if } \frac{1}{SP'^2} - \frac{1}{(SP_2 + SP')^2} = \frac{SP_1}{SP'^3};$$

or, dropping the suffixes, because $SP_1 = SP_2$,

$$(SP + SP')^2 - SP'^2 = \frac{SP}{SP'} (SP + SP')^2,$$

$$\text{or } SP.SP' + 2SP'^2 = SP^2 + 2SP.SP'^2 + SP'^2,$$

$$\text{or } SP'^2 = SP(SP + SP'),$$

or if SP' be a mean proportional between SP and $SP + SP'$.

1849.

If, in addition to the force of the Sun on a planet, there be a small force tending towards the Sun, and varying inversely as the m^{th} power of the distance of the planet from the Sun, prove that the perihelion of the orbit will have a progressive or regressive motion, according as m is greater or less than 2.

Can you explain this result by reasoning similar to that used in "Airy's *Gravitation*"?

If P be the whole central force on the planet we shall have

$$P = \mu u^2 + \mu' u^m,$$

where μ' is very small. The equation of motion is

$$\frac{d^2 u}{d\theta^2} + u - \frac{P}{h^2 u^2} = 0,$$

$$\text{or } \frac{d^2 u}{d\theta^2} + u - \frac{\mu}{h^2} - \frac{\mu'}{h^2} u^{m-2} = 0.$$

For a first approximation,

$$\frac{d^2 u}{d\theta^2} + u - \frac{\mu}{h^2} = 0,$$

which will be very approximately satisfied by

$$u = a \{1 + e \cos(c\theta - \alpha)\},$$

if e be very near unity, and $a = \frac{\mu}{h^2}$;

$$\therefore \frac{\mu'}{h^2} u^{m-1} = \frac{\mu'}{\mu} a^{m-2} \{1 + (m-2) e \cos(c\theta - \alpha)\},$$

omitting higher powers of e .

Hence, for a second approximation,

$$\frac{d^2 u}{d\theta^2} + u - a - \frac{\mu'}{\mu} a^{m-1} - \frac{\mu'}{\mu} a^{m-1} (m-2) e \cos(c\theta - \alpha) = 0,$$

which is satisfied by

$$u = a \left\{ 1 + \frac{\mu'}{\mu} a^{m-2} + e \cos(c\theta - \alpha) \right\},$$

$$\text{if } ae(1 - c^2) \cos(c\theta - \alpha) - \frac{\mu'}{\mu} a^{m-1} (m-2) e \cos(c\theta - \alpha) = 0,$$

$$\text{or } 1 - c^2 = \frac{\mu'}{\mu} (m-2) a^{m-2};$$

hence c is $<$ or $>$ 1 according as m is $>$ or $<$ 2.

Now, the argument $(c\theta - \alpha)$ may be put in the form

$$\theta - \{\alpha + (1 - c) \theta\};$$

whence it appears that the above equation between u and θ is the equation to an ellipse, the longitude of whose apse is $\alpha + (1 - c) \theta$; its apse will therefore progress or regress according as c is $<$ or $>$ 1, *i.e.* according as m is $>$ or $<$ 2.

This result may be explained in a manner similar to that used in Airy's *Gravitation*, Art. 98.

Let P , A be perihelion and aphelion.

The disturbing force is towards S both at P and A ; it will therefore progress about P and regress about A . To consider which of these effects will be the greater. If the disturbing force at P , A and the other points of the orbit were proportional to the inverse square of the distance, its only effect would be to alter the magnitude of the central force in a certain ratio without altering its law; it would therefore have no effect upon the position of the apsides, or its effects about P and A would be equal. But if the disturbing force vary inversely as the (distance) ^{m} , where m is $>$ 2, the ratio of its intensity at P to its intensity at A will be greater than the ratio of the intensities of the central force at those points; hence its effect will be greater at P than at A , or the progression at P will be greater than the regression at A ; *i.e.* on the whole the perihelion will progress. Similarly, it may be shewn that if m be $<$ 2, the perihelion will regress.

ATTRACTIONS.

1848.

1. A sphere is composed of an immense number of free particles, equally distributed, which gravitate to each other without interfering: supposing the particles to have no initial velocity, prove that the mean density about a given particle will vary inversely as the cube of its distance from the centre.

The attraction upon any particle will be the same as if the matter nearer than itself to the centre were collected there, and attracted with a force varying inversely as the square of the distance. This attracting mass will remain the same for the same particle throughout the motion. Let x , $x + \delta x$ be the distances from the centre at the time t , of two particles situated in the same radius, whose original distances from the centre were a , $a + \delta a$;

$$\therefore \frac{d^2x}{dt^2} = -\frac{\mu}{x^2},$$

$$\text{and } \left(\frac{dx}{dt}\right)^2 = 2\mu \left(\frac{1}{x} - \frac{1}{a}\right) = 2\mu \frac{a-x}{ax};$$

$$\begin{aligned} \therefore \frac{dt}{dx} &= -\left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \frac{x}{(ax-x^2)^{\frac{1}{2}}} \\ &= -\left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \left\{ \frac{\frac{1}{2}a}{(ax-x^2)^{\frac{1}{2}}} - \frac{\frac{1}{2}a-x}{(ax-x^2)^{\frac{1}{2}}} \right\}; \end{aligned}$$

$$\therefore t = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \left\{ \frac{a}{2} \left(\frac{\pi}{2} + \sin^{-1} \frac{\frac{1}{2}a-x}{\frac{1}{2}a} \right) + (ax-x^2)^{\frac{1}{2}} \right\}.$$

But μ depends upon the mass originally contained within the sphere radius a ;

$$\therefore \mu \propto a^3 = Ca^3 \text{ suppose};$$

$$\begin{aligned}\therefore t &= \frac{1}{(2C)^{\frac{1}{2}}} \left\{ \frac{\pi}{4} + \frac{1}{2} \sin^{-1} \left(1 - \frac{2x}{a} \right) + \left(\frac{x}{a} - \frac{x^2}{a^2} \right)^{\frac{1}{2}} \right\} \\ &= f \left(\frac{x}{a} \right).\end{aligned}$$

In order to find the relation between δx at the time t and δa , we must differentiate this equation, considering x and a variable, and t constant;

$$\therefore 0 = f' \left(\frac{x}{a} \right) \frac{a\delta x - x\delta a}{a^2},$$

$$\text{or } \delta x = \frac{x}{a} \delta a.$$

Hence the volume of the shell originally contained between the spheres of radii a , $a + \delta a$, *i.e.* of volume $4\pi a^2 \delta a$, is now of volume $4\pi x^2 \delta x = 4\pi \frac{x^3}{a} \delta a \propto x^3$. Hence the density of the matter in this shell, which varies inversely as the volume, varies inversely as its (radius)³: hence the proposition is true.

2. Prove geometrically, or otherwise, that if g be the attraction which a particle m exerts on a point in a closed surface S , θ the angle between the direction of g and the normal, $d\omega$ an element of S ,

$$\iint g \cos \theta d\omega = 4\pi m, \quad \text{or} = 0,$$

according as m is within or without S , the attraction of m at the distance r being $\frac{m}{r^2}$.

Extend this result to the case of a finite mass cut by S , and thence prove by taking for S an elementary parallelopiped, that if V be the potential of any mass for an internal particle,

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = -4\pi\rho.$$

About the particle m as centre describe a sphere of radius unity; and let a cone having m in its vertex, and circumscribing the element $d\omega$ of the surface S , include a portion $d\omega'$ of the surface of this sphere: then the relation between $d\omega$ and $d\omega'$ will be

$$d\omega \cos \theta = r^2 \cdot d\omega'.$$

Also, let g' be the attraction which m exerts on a point at distance unity;

$$\therefore g = \frac{g'}{r^2};$$

$$\text{hence } g \cos \theta d\omega = g' d\omega',$$

$$\text{and } \iint g \cos \theta d\omega = g' \omega'$$

$$= \text{the whole attraction of } m \text{ on } \omega',$$

where ω' is the whole projection of S on the surface of the sphere: hence, if m be external to S we see, by taking the projection of each element with its proper sign, that $\omega' = 0$; but if m be within S , $\omega' = 4\pi$; and, by the question, $g' = m$;

$$\therefore \iint g \cos \theta d\omega = 4\pi m, \text{ or } = 0,$$

according as m is within S or without it.

This equation expresses the value of the sum of the attractions of a particle m on the different points of a closed surface, each resolved in the normal to the surface at the point.

Now, suppose the surface S to cut from a finite mass the mass M , the above equation holds for every element of this mass, and therefore for the whole, if the symbols involved be properly modified: we shall, therefore, still have the sum of the attractions on each point of S , resolved in the normal at that point, $= 4\pi M$.

Again, suppose S to be an elementary parallelopiped so small that the density (ρ) may be supposed uniform throughout it: let V be the potential of a mass for an internal particle whose coordinates are x, y, z . Let P (fig. 121) be the point x, y, z , and the corner of a parallelopiped whose edges $\delta x, \delta y, \delta z$, are parallel to the coordinate axes.

The above considerations shew that the sum of the attractions on the faces, each resolved in a direction perpendicular to the face, will be due to the matter contained in the parallelopiped: now

$$\frac{dV}{dx} = - \iiint \frac{\rho(x-\xi) d\xi d\eta d\zeta}{\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}^{\frac{3}{2}}},$$

the integration extending throughout the parallelopiped; hence at P , $\frac{dV}{dx}$ will be positive, since ξ is always greater than x ; at P it will be negative, since ξ is always less than $x + \delta x$. Hence the absolute magnitude of the attractions parallel to the axis of x at P and p will be, respectively,

$$\frac{dV}{dx} \text{ and } -\frac{dV}{dx} - \frac{d^2V}{dx^2} \delta x.$$

Now we may consider the surfaces MP , mp so small, that the attraction on every point of each of them is the same: hence the whole attractions on MP , mp parallel to the axis of x

$$\begin{aligned} &= \frac{dV}{dx} \delta y \delta z + \left(-\frac{dV}{dx} - \frac{d^2V}{dx^2} \delta x \right) \delta y \delta z \\ &= -\frac{d^2V}{dx^2} \delta x \delta y \delta z. \end{aligned}$$

The whole expression for $\iint g \cos \theta d\omega$ is in this way found to be

$$\begin{aligned} &\left(-\frac{d^2V}{dx^2} - \frac{d^2V}{dy^2} - \frac{d^2V}{dz^2} \right) \delta x \delta y \delta z, \text{ which } \therefore = 4\pi M \\ &= 4\pi \rho \delta x \delta y \delta z; \\ \therefore \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} &= -4\pi \rho. \end{aligned}$$

3. Supposing a mass of homogeneous fluid, which attracts every particle of matter with a force varying as $\frac{1}{(\text{dist.})^2}$, to be enclosed within a thin spherical shell, find the path described by a heavy body let fall from any point of the surface of the fluid, the resistance varying as the velocity. Prove also that the body will reach the axis and equator of the spheroid after the same intervals respectively, from whatever points of the surface it begins to fall.

The attractions of the spheroid on any particle within it perpendicular to the axis and equator, vary respectively as the distances (x, y) of the particle from that line and plane, $= \mu x, \mu' x$ suppose.

Also, by the question, the resistance on the particle in motion
 $= kv = k \frac{ds}{dt}$: hence the resolved parts of it are

$$k \frac{ds}{dt} \cdot \frac{dx}{ds}, \quad k \frac{ds}{dt} \cdot \frac{dy}{ds}, \quad \text{or} \quad k \frac{dx}{dt}, \quad k \frac{dy}{dt}.$$

Hence the equations of motion are

$$\frac{d^2x}{dt^2} + k \frac{dx}{dt} + \mu x = 0,$$

$$\frac{d^2y}{dt^2} + k \frac{dy}{dt} + \mu'y = 0.$$

Let α, β be the roots of the equation

$$z^2 + kz + \mu = 0,$$

and α', β' those of

$$z^2 + kz + \mu' = 0:$$

the above equations give

$$x = A\varepsilon^{\alpha t} + B\varepsilon^{\beta t},$$

$$y = A'\varepsilon^{\alpha' t} + B'\varepsilon^{\beta' t},$$

A, B, A', B' , being arbitrary constants to be determined by the circumstance that the body falls from rest from a given position.

The circumstance that it falls from rest gives us the condition

that $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0$, when $t = 0$;

$$\therefore 0 = A\alpha + B\beta,$$

$$\text{and } 0 = A'\alpha' + B'\beta';$$

$$\therefore \frac{A}{\beta} = -\frac{B}{\alpha} = C \text{ suppose,}$$

$$\frac{A'}{\beta'} = -\frac{B'}{\alpha'} = C' \text{ suppose;}$$

$$\therefore x = C(\beta\varepsilon^{\alpha t} - \alpha\varepsilon^{\beta t}),$$

$$y = C'(\beta'\varepsilon^{\alpha' t} - \alpha'\varepsilon^{\beta' t}),$$

the equations for the determination of the relation between x and y by the elimination of t .

Hence, if t, t' be times of falling to the axis and equator respectively,

$$0 = \beta\varepsilon^{\alpha t} - \alpha\varepsilon^{\beta t},$$

$$\therefore \log \beta + \alpha t = \log \alpha + \beta t;$$

$$\therefore t = \frac{\log \alpha - \log \beta}{\alpha - \beta},$$

$$\text{so } t' = \frac{\log \alpha' - \log \beta'}{\alpha' - \beta'};$$

whence it appears that t, t' are independent of the particle's original position.

1849.

1. Each particle of two indefinite straight lines, lying in the same plane, attracts with a force which varies inversely as the distance. Determine the motion of a body projected in any direction along the plane.

We must first find the attraction of either of the lines AB (fig. 122) upon the particle in any position P . From P draw PD , the perpendicular on AB , and join PQ , Q being a point at the distance x from D . Let δA be the attraction of an element δx of the line about Q resolved in PD :

$$\therefore \delta A = \frac{\mu dx}{PQ} \cos QPD$$

$$= \frac{\mu a dx}{a^2 + x^2};$$

$$\therefore A = \mu \tan^{-1} \frac{x}{a}$$

$$\left. \begin{array}{l} \text{from } x = -\infty \\ \text{to } x = +\infty \end{array} \right\} = \mu\pi.$$

Hence P will be attracted by two constant attractions in constant directions, which are therefore equivalent to a constant attraction in a constant direction, viz. $2\mu\pi \sin \alpha$ (2α the angle between the lines), parallel to the internal bisector of the lines. Hence the case is the common case of projectiles, and the path will be parabolic.

2. The attraction of a uniform filament of matter, in the form of a plane curve, upon a particle is replaced by that of a circular filament having the particle for its centre: find the

law of density of the circular filament in order that this may be done.

Let the curve be referred to the particle as pole, and let a be the radius of the circle, μ the density of the filament in the form of the curve, ρ that of the circular filament at the point (θ) ;

$$\begin{aligned}\therefore \frac{\rho a \delta \theta}{a^2} &= \frac{\mu \delta s}{r^2}; \\ \therefore \rho &= \frac{\mu a}{r^2} \frac{ds}{d\theta} \\ &= \frac{\mu a}{r^2} \left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} \\ &= \mu a \left\{ u^2 + \left(\frac{du}{d\theta} \right)^2 \right\}^{\frac{1}{2}}, \quad u = \frac{1}{r}, \\ &= \frac{\mu a}{p},\end{aligned}$$

if p be the perpendicular from the particle on the tangent at the point (r, θ) ;

$$\therefore \rho \propto \frac{1}{p}.$$

1850.

A uniform rod is placed with its middle point against a rough circle, in whose centre resides a force attracting inversely as the square of the distance: if the rod be slightly disturbed from the position of equilibrium, find the time of a small oscillation.

Let θ be the inclination of the rod to the horizon at the time t ; ξ the distance of its middle point from the point of contact with the circle. Since the motion is small, we may take the equations of motion about the instantaneous axis of rotation: hence we have

$$M(k^2 + \xi^2) \frac{d^2 \theta}{dt^2} = -L,$$

where k is the radius of gyration of the rod about its middle point, and L the moment of the attractions on the rod about the point of contact.

To find L . In fig. 120 the moment about D of the attraction on an element at Q (P being the centre of the circle),

$$\delta L = \frac{\mu \rho \delta x}{PQ^2} \cos QPD \cdot QD,$$

if μ be the absolute force of the attraction, ρ the mass of a unit of length of the rod,

$$= \frac{\mu \rho a x \delta x}{(a^2 + x^2)^{\frac{3}{2}}};$$

$$\therefore L = \mu \rho a \left\{ C - \frac{1}{(a^2 + x^2)^{\frac{1}{2}}} \right\},$$

from $x = -(l - \xi)$
to $x = l + \xi$ $\left. \vphantom{\int} \right\}$, $2l$ being the length of the rod,

$$\begin{aligned} &= \mu \rho a \left\{ \frac{1}{\{a^2 + (l - \xi)^2\}^{\frac{1}{2}}} - \frac{1}{\{a^2 + (l + \xi)^2\}^{\frac{1}{2}}} \right\} \\ &= \frac{\mu \rho a}{(a^2 + l^2)^{\frac{1}{2}}} \left\{ 1 + \frac{l\xi}{a^2 + l^2} - \left(1 - \frac{l\xi}{a^2 + l^2} \right) \right\} : \end{aligned}$$

omitting ξ^2 and higher powers of ξ ,

$$= \frac{2\mu \rho a l}{(a^2 + l^2)^{\frac{3}{2}}} \xi = \frac{2\mu \rho a^2 l}{(a^2 + l^2)^{\frac{3}{2}}} \theta,$$

to the same degree of approximation: and the equation of motion becomes

$$Mk^2 \frac{d^2\theta}{dt^2} + \frac{2\mu \rho a^2 l}{(a^2 + l^2)^{\frac{3}{2}}} \theta = 0,$$

or, since $M = 2l\rho$,

$$\frac{d^2\theta}{dt^2} + \frac{\mu a^2}{(a^2 + l^2)^{\frac{3}{2}} k^2} \theta = 0.$$

Hence the time of a small oscillation

$$T = 2\pi \frac{k(a^2 + l^2)^{\frac{3}{4}}}{\mu^{\frac{1}{2}} a}.$$

1851.

1. Two uniform straight rods AB , CD (fig. 123), mutually attracting each other with forces varying as the distance, are

constrained to move in two grooves ABO , CDO at right angles to each other; determine the time at which the extremity of one of the rods reaches O the point of intersection of the grooves.

The attraction towards O of each element $\rho'\delta\eta$ of CD upon any element $\rho\delta\xi$ of AB , at a distance ξ from O , will be the same, viz. $\mu\rho'\delta\eta\rho\delta\xi.\xi$: hence, if $AB=2a$, $CD=2b$, the whole attraction towards O of CD on AB will

$$= 2\mu\rho\rho'b \int_{x-a}^{x+a} \xi d\xi:$$

if x be the distance from O of the middle point of AB ,

$$= 4\mu\rho\rho'box.$$

The equation of motion of AB is therefore

$$2\rho a \frac{d^2x}{dt^2} = -4\mu\rho\rho'box,$$

$$\text{or } \frac{d^2x}{dt^2} + n^2x = 0, \text{ if } n^2 = 2\mu\rho'b;$$

$$\therefore x = A \cos(nt + B)$$

$$= x_0 \cos nt,$$

if $t = 0$ at the beginning of motion, and x_0 is the original value of x .

Hence, if t be the interval before A arrives at O , we have

$$a = x_0 \cos nt,$$

$$\text{or } t = \frac{1}{(2\mu\rho'b)^{\frac{1}{2}}} \cos^{-1} \frac{a}{x_0}.$$

2. If a portion of a thin spherical shell, whose projections upon the three coordinate planes through the centre are A , B , C , attract a particle at the centre with a force varying as any function of the distance, shew that the particle will begin to move in the direction of a straight line whose equations are

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C}.$$

Let θ be the angle which the radius drawn to the element δS of the shell makes with the axis of x ; then, if $r =$ radius

of sphere, and $\phi(r)$ be the law of attraction, the attraction of δS on the particle parallel to the axis of x will be

$$\phi(r) \delta S \cos \theta,$$

and the whole attraction on it (X) parallel to the axis of x ,

$$\begin{aligned} X &= \phi(r) \Sigma. \delta S \cos \theta, \quad \text{since } r \text{ is constant,} \\ &= \phi(r) A. \end{aligned}$$

$$\text{So } Y = \phi(r) B,$$

$$Z = \phi(r) C.$$

And the equations to the direction of the resultant attraction, which is the direction in which the particle will begin to move, are

$$\frac{x}{X} = \frac{y}{Y} = \frac{z}{Z},$$

$$\text{or } \frac{x}{A} = \frac{y}{B} = \frac{z}{C}.$$

PHYSICAL OPTICS.

1848.

A spherical wave of light is incident directly on a lens: find approximately the retardation of the several portions of the wave, and prove in this way the common equation $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$.

Suppose the lens to be a positive concavo-convex whose thickness at the middle point is indefinitely small: take this middle point as origin of coordinates and the axis of the lens for axis of x . The retardation of any part or ray of the wave will $= (\mu - 1) \times$ length of the path in glass $= (\mu - 1) \rho$ suppose.

Let x, y be the coordinates of the point of incidence of this ray, θ the inclination to the axis of the part of the ray within the lens.

The equation to the two surfaces will be

$$\eta^2 = 2r\xi \dots\dots\dots (1),$$

$$\text{and } \eta^2 = 2s\xi \dots\dots\dots (2),$$

very nearly; since ξ is very small for all rays near the axis.

Now (2) is satisfied by the coordinates $x - \rho \cos \theta, y - \rho \sin \theta$, or, as θ is very small as well as ρ , by $x - \rho$ and y ;

$$\therefore y^2 = 2s(x - \rho) = \frac{s}{r} y^2 - 2s\rho;$$

$$\therefore \rho = \left(\frac{1}{r} - \frac{1}{s} \right) \frac{y^2}{2};$$

therefore the retardation of this portion of the wave

$$= (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right) \frac{y^2}{2}.$$

Hence, if u be the distance from the origin of the centre of the incident wave, the equivalent length in air of the ray

we are considering from the centre of the wave to the point of emergence,

$$\begin{aligned}
 &= \{(u-x)^2 + y^2\}^{\frac{1}{2}} + \mu\rho \\
 &= (u-x) \left\{ 1 + \frac{y^2}{2(u-x)^2} \right\} + \mu\rho \\
 &= u-x + \frac{y^2}{2u} + \mu \left(\frac{1}{r} - \frac{1}{s} \right) \frac{y^2}{2} \text{ very nearly} \\
 &= u + \left\{ \frac{1}{u} - \frac{1}{r} + \mu \left(\frac{1}{r} - \frac{1}{s} \right) \right\} \frac{y^2}{2} \text{ by (1).}
 \end{aligned}$$

Consequently this ray upon emergence is in the same phase as the ray incident directly when it has travelled a distance $\left\{ \frac{1}{u} - \frac{1}{r} + \mu \left(\frac{1}{r} - \frac{1}{s} \right) \right\} \frac{y^2}{2}$ after emergence.

Now the geometrical focus upon emergence is the centre of curvature at the vertex of the surface of revolution, which is the locus of all parts of the wave which, after transmission, are in the same phase of vibration.

Let v be the radius of this sphere when only the parts of the wave indefinitely near the axis have emerged: the sphere will then pass through the points

$$(x-\rho, y) \text{ or } \left(\frac{y^2}{2s}, y \right) \text{ and } \left[- \left\{ \frac{1}{u} - \frac{1}{r} + \mu \left(\frac{1}{r} - \frac{1}{s} \right) \right\} \frac{y^2}{2}, 0 \right],$$

y being indefinitely small;

$$\therefore y^2 = 2v \left[\left\{ \frac{1}{u} - \frac{1}{r} + \mu \left(\frac{1}{r} - \frac{1}{s} \right) \right\} \frac{y^2}{2} + \frac{y^2}{2s} \right];$$

$$\therefore \frac{1}{v} = \frac{1}{u} + (\mu-1) \left(\frac{1}{r} - \frac{1}{s} \right),$$

$$\text{or } \frac{1}{v} - \frac{1}{u} = \frac{1}{f}, \quad \text{if } \frac{1}{f} = (\mu-1) \left(\frac{1}{r} - \frac{1}{s} \right).$$

But evidently v is the distance from the lens of the geometrical focus upon emergence; hence this is the usual formula.

1850.

A and B being two fixed points, and P such that $AP = \mu.BP$, the locus of P is a circle. Shew from this property how to construct a lens of common glass, such that a direct pencil incident from a determinate point will be refracted without aberration.

The property enunciated will be found in Prob. 8, p. 157.

Let A, B (fig. 124) be the points from which the pencil is to diverge before and after passing through the lens without aberration. Draw the circular arc HCH' such, that if Q be any point in it, $BQ = \mu.AQ$.

With centre A describe any circular arc HcH' intersecting HCH' in H, H' : $HCH'c$ is the section by the plane of the paper of such a lens as is required. For a ray incident upon the lens from A will suffer no deviation; and

$$\begin{aligned} BQ - (\mu.QP + AP) &= \mu.AQ - (\mu.QP + AP) \\ &= (\mu - 1)AP \text{ is constant:} \end{aligned}$$

and therefore, by reasoning similar to that in Airy's *Tracts*, p. 276, it appears that the pencil diverging from A will, after emergence, diverge from B .

1851.

If (θ) be the angle which one of the planes of polarization makes with the plane passing through the normal to the front of the wave and either optic axis of a biaxial crystal, and v_1, v_2 be the two velocities of transmission of the wave, shew that

$$(v_1 \cos \theta)^2 + (v_2 \sin \theta)^2 = b^2.$$

Since the planes of polarization respectively bisect the acute and obtuse angles between the two planes through the normal to the front and the optic axes (Griffin's *Double Refraction*, Art. 21, p. 12), it follows that the angle between these two planes $= 2\theta$.

Now, in accordance with the usual notation, the equations to normal to the plane front are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \dots\dots\dots(1).$$

Those to the optic axes are

$$y = 0, \quad \frac{x}{(a^2 - b^2)^{\frac{1}{2}}} \pm \frac{z}{(b^2 - c^2)^{\frac{1}{2}}} = 0 \dots\dots\dots(2).$$

If therefore the equations to the above planes are

$$Ax + By + Cz = 0,$$

$$A'x + B'y + C'z = 0,$$

we must have

$$Al + Bm + Cn = 0,$$

$$A(a^2 - b^2)^{\frac{1}{2}} + C(b^2 - c^2)^{\frac{1}{2}} = 0;$$

$$\therefore \frac{A}{m(b^2 - c^2)^{\frac{1}{2}}} = \frac{B}{n(a^2 - b^2)^{\frac{1}{2}} - l(b^2 - c^2)^{\frac{1}{2}}} = -\frac{C}{m(a^2 - b^2)^{\frac{1}{2}}} = r \text{ suppose.}$$

Similarly

$$\frac{A'}{m(b^2 - c^2)^{\frac{1}{2}}} = -\frac{B'}{n(a^2 - b^2)^{\frac{1}{2}} + l(b^2 - c^2)^{\frac{1}{2}}} = \frac{C'}{m(a^2 - b^2)^{\frac{1}{2}}} = r' \text{ suppose.}$$

$$\begin{aligned} \text{Also } \cos 2\theta &= \frac{AA' + BB' + CC'}{(A^2 + B^2 + C^2)^{\frac{1}{2}}(A'^2 + B'^2 + C'^2)^{\frac{1}{2}}} \\ &= \frac{AA' + BB' + CC'}{\{(AA' + BB' + CC')^2 + (BC' - CB')^2 + (CA' - AC')^2 + (AB' - BA')^2\}^{\frac{1}{2}}}. \end{aligned}$$

Now

$$\begin{aligned} AA' + BB' + CC' &= \{m^2(b^2 - c^2) + l^2(b^2 - c^2) - n^2(a^2 - b^2) - m^2(a^2 - b^2)\} rr' \\ &= \{(l^2 + m^2)(b^2 - c^2) - (m^2 + n^2)(a^2 - b^2)\} rr' \\ &= \{(l^2 + m^2)b^2 - (1 - n^2)c^2 - (1 - l^2)a^2 + (m^2 + n^2)b^2\} rr' \\ &\quad \because l^2 + m^2 + n^2 = 1 \\ &= -\{a^2 - b^2 + c^2 - (l^2a^2 + m^2b^2 + n^2c^2)\} rr' \\ &= -(U - 2b^2) rr' \text{ suppose,} \\ (BC' - CB')^2 + (CA' - AC')^2 + (AB' - BA')^2 \\ &= 4(m^2l^2 + m^4 + m^2n^2)(a^2 - b^2)(b^2 - c^2)r^2r'^2 \\ &= 4m^2(a^2 - b^2)(b^2 - c^2)r^2r'^2 \\ &= 4Vr^2r'^2 \text{ suppose;} \end{aligned}$$

$$\begin{aligned}\therefore \cos 2\theta &= \frac{2b^2 - U}{\{(2b^2 - U)^2 + 4V\}^{\frac{1}{2}}} \\ &= \frac{2b^2 - U}{[U^2 - 4\{b^2(U - b^2) - V\}]^{\frac{1}{2}}},\end{aligned}$$

$$\begin{aligned}b^2(U - b^2) - V &= b^2\{a^2 + c^2 - (l^2a^2 + m^2b^2 + n^2c^2)\} - m^2(a^2 - b^2)(b^2 - c^2) \\ &= (1 - n^2 - m^2)b^2c^2 + m^2c^2a^2 + (1 - l^2 - m^2)a^2b^2 \\ &= l^2b^2c^2 + m^2c^2a^2 + n^2a^2b^2 \\ &= W \text{ suppose};\end{aligned}$$

$$\therefore \cos 2\theta = \frac{2b^2 - U}{(U^2 - 4W)^{\frac{1}{2}}}.$$

Now v_1^2, v_2^2 are the roots of the equation

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0,$$

considered as a quadratic in v^2 ;

$$\begin{aligned}\therefore v_1^2 + v_2^2 &= l^2(b^2 + c^2) + m^2(c^2 + a^2) + n^2(a^2 + b^2) \\ &= (1 - m^2 - n^2)(b^2 + c^2) + (1 - n^2 - l^2)(c^2 + a^2) + (1 - l^2 - m^2)(a^2 + b^2) \\ &= a^2 + b^2 + c^2 - (l^2a^2 + m^2b^2 + n^2c^2) \\ &= U,\end{aligned}$$

$$\begin{aligned}\text{and } v_1^2 v_2^2 &= l^2b^2c^2 + m^2c^2a^2 + n^2a^2b^2 \\ &= W;\end{aligned}$$

$$\begin{aligned}\therefore \cos 2\theta &= \frac{2b^2 - v_1^2 - v_2^2}{(v_1^2 + v_2^2 - 4v_1^2 v_2^2)^{\frac{1}{2}}} \\ &= \frac{2b^2 - v_1^2 - v_2^2}{v_1^2 - v_2^2};\end{aligned}$$

$$\therefore v_1^2(1 + \cos 2\theta) + v_2^2(1 - \cos 2\theta) = 2b^2,$$

$$\text{and } (v_1 \cos \theta)^2 + (v_2 \sin \theta)^2 = b^2.$$

CALCULUS OF VARIATIONS.

1848.

A and B are two given points in the generating line of a surface of revolution whose axis is vertical: supposing a body acted on by gravity, to descend along the surface from A to B , find its form when the whole pressure upon it between the two given points is the least possible.

Find also the form of the surface when the length of the generating line between the point A and B is also given, and point out the difference between the two results.

Let y be the depth of any point of AB below A , x its distance from the axis; the pressure at this point will be

$$P = Mg \frac{dx}{ds} + M \frac{v^2}{\rho},$$

v being the velocity, and ρ the radius of curvature at the point: hence we must have

$$\int P ds = Mg \int \left(1 + \frac{2y}{\rho} \cdot \frac{ds}{dx} \right) dx, \text{ a minimum.}$$

$$\text{Here } V = 1 + \frac{2y}{\rho} \frac{ds}{dx}$$

$$= 1 - \frac{2qy}{1+p^2};$$

$$\therefore N = \frac{dV}{dy} = - \frac{2q}{1+p^2},$$

$$P = \frac{dV}{dp} = \frac{4qpy}{(1+p^2)^2},$$

$$Q = \frac{dV}{dq} = - \frac{2y}{1+p^2};$$

the equation between N , P , Q , is

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} = 0,$$

$$\text{or } N - \frac{d}{dx} \left(P - \frac{dQ}{dx} \right) = 0.$$

$$\text{But } \frac{dQ}{dx} = -\frac{2p}{1+p^2} + \frac{4qpy}{(1+p^2)^2};$$

$$\therefore P - \frac{dQ}{dx} = \frac{2p}{1+p^2} = f(p);$$

$$\therefore N - f'(p) \cdot q = 0$$

$$- \frac{2q}{1+p^2} - f'(p) \cdot q = 0,$$

$$\text{or } q = 0,$$

shewing that the line required is the straight line joining the points A , B .

If the length of the line be given, we have

$$V = 1 + \frac{2y}{\rho} \frac{ds}{dx} + a(1+p^2)^{\frac{1}{2}},$$

a being an arbitrary constant to be determined.

N and Q remain of the same value as before; P becomes

$$P = \frac{4qpy}{(1+p^2)^2} + \frac{ap}{(1+p^2)^{\frac{1}{2}}};$$

$$\therefore P - \frac{dQ}{dx} = \frac{ap}{(1+p^2)^{\frac{1}{2}}} + f(p).$$

Hence we shall find, as before, that $q = 0$, or the required curve is a straight line: in this case, however, it must be a broken line, the different parts of which are equally inclined to the vertical, and the inclination so chosen as to give the line of the required length. The particle is of course supposed to turn the abrupt angles of the line without impulsive pressure or change of velocity.

1851.

A uniform straight rod AB is constrained to move in a vertical plane with its middle point in a horizontal groove, and its upper extremity against a smooth curve: find the nature of the curve when the rod descends from one given position to another in the least time possible, the initial angular velocity being given.

Let CB (fig. 125) be the required curve, OQ the horizontal groove; take the point O in it for origin of coordinates: at time t let AB be the position of the rod, draw BT the tangent at P ; $ON = x$, $NB = y$, $OQ = \xi$, $\angle OQB = \theta$, $QB = a$.

The equation of *vis viva* is

$$\left(\frac{d\xi}{dt}\right)^2 + k^2 \left(\frac{d\theta}{dt}\right)^2 = \text{constant} = c^2 \text{ suppose} \dots\dots(1).$$

Also the motion of B perpendicular to BT is zero,

$$\therefore \frac{d\xi}{dt} \sin BTN + a \frac{d\theta}{dt} \cos QBT = 0,$$

$$\text{or, if } BTN = \phi, \quad \frac{d\xi}{dt} \sin \phi + a \frac{d\theta}{dt} \cos(\phi - \theta) = 0 \dots\dots(2);$$

$$\text{also } (\xi + x)^2 + y^2 = a^2 \dots\dots\dots(3).$$

From (1) and (2)

$$\left\{1 + \frac{k^2 \sin^2 \phi}{a^2 \cos^2(\phi - \theta)}\right\} \left(\frac{d\xi}{dt}\right)^2 = c^2,$$

$$\text{or, since } \sin \theta = \frac{y}{a}, \quad \tan \phi = \frac{dy}{dx} = p,$$

$$\left[1 + \frac{k^2 p^2}{\{(a^2 - y^2)^{\frac{1}{2}} + py\}^2}\right] \left(\frac{d\xi}{dt}\right)^2 = c^2.$$

Also, from (3)

$$\xi + x = (a^2 - y^2)^{\frac{1}{2}};$$

$$\therefore \frac{d\xi}{dx} + 1 = - \frac{py}{(a^2 - y^2)^{\frac{1}{2}}};$$

$$\therefore \frac{d\xi}{dx} = - \frac{(a^2 - y^2)^{\frac{1}{2}} + py}{(a^2 - y^2)^{\frac{1}{2}}} ;$$

$$\begin{aligned} \therefore dt &= \frac{1}{c} \left[1 + \frac{k^2 p^2}{\{(a^2 - y^2)^{\frac{1}{2}} + py\}^2} \right]^{\frac{1}{2}} d\xi \\ &= - \frac{1}{c(a^2 - y^2)^{\frac{1}{2}}} [\{(a^2 - y^2)^{\frac{1}{2}} + py\}^2 + k^2 p^2]^{\frac{1}{2}} dx. \end{aligned}$$

Here V contains only y and p ,

$$\therefore V - Pp = C,$$

$$\text{or } [\{(a^2 - y^2)^{\frac{1}{2}} + py\}^2 + k^2 p^2]^{\frac{1}{2}} - \frac{py \{ (a^2 - y^2)^{\frac{1}{2}} + py \} + k^2 p^2}{[\{(a^2 - y^2)^{\frac{1}{2}} + py\}^2 + k^2 p^2]^{\frac{1}{2}}} + \frac{C}{c} (a^2 - y^2)^{\frac{1}{2}} = 0;$$

$$\therefore (a^2 - y^2)^{\frac{1}{2}} \{ (a^2 - y^2)^{\frac{1}{2}} + py \} + \frac{C}{c} (a^2 - y^2)^{\frac{1}{2}} [\{(a^2 - y^2)^{\frac{1}{2}} + py\}^2 + k^2 p^2]^{\frac{1}{2}} = 0;$$

$$\therefore \left\{ 1 - \left(\frac{C}{c} \right)^2 \right\}^{\frac{1}{2}} \{ (a^2 - y^2)^{\frac{1}{2}} + py \} = kp,$$

$$\text{or } (C'k - y)p = (a^2 - y^2)^{\frac{1}{2}}, \quad C' = \left\{ 1 - \left(\frac{C}{c} \right)^2 \right\}^{-\frac{1}{2}};$$

$$\therefore C'k \sin^{-1} \frac{y}{a} + (a^2 - y^2)^{\frac{1}{2}} = x + C'',$$

the required equation to the curve: the constants C' , C'' are to be determined by the two given positions of the rod which give two points through which the curve must pass. The curve is independent of the angular velocity of projection.

APPENDIX.

THE following problem in Astronomy was set in 1848.

If a rectangular court be enclosed within a wall of given height, and one of its sides be inclined at an angle of 30° to the meridian, determine the breadths of the shadows of the walls on a given day at noon, and the portions of the courts and walls which will be enveloped in the shadow, the latitude being $52^\circ 30'$ north, and the Sun's declination on the given day $7^\circ 30'$ north.

By referring to the problem on p. 64, we see that here $\theta = 30^\circ$ and $\phi = \text{latitude} - \text{Sun's declination} = 45^\circ$,

$$\therefore \frac{a}{b} = \tan \theta = \frac{1}{3^{\frac{1}{2}}}, \quad \frac{h}{(a^2 + b^2)^{\frac{1}{2}}} = \tan \phi = 1;$$

$$\therefore a = \frac{1}{2}h, \quad b = \frac{3^{\frac{1}{2}}}{2}h.$$

Let l_1, l_2 be the lengths of the walls, whose shadows are respectively of the breadth a, b , the area of the courts enveloped in shade will be $l_1a + (l_2 - a)b$, or $l_1a + l_2b - ab$; and the shaded parts of the walls the whole of the two walls, and two triangles $\frac{1}{2}ha, \frac{1}{2}hb$ of the other two.

The following solution of the problem on p. 148, is due to Mr. Gaskin.

Let TP, TQ (fig. 126) be the two given tangents, take the line AB as axis of x , and let $OP'Q'$ be the chord of contact of any conic touching TP, TQ , and passing through A, B . Take

O as origin, and let

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a'} + \frac{y}{b'} = 1$$

be the equations to TP , TQ respectively; also let $OA = \alpha$, $OB = \beta$. Let the equation to $OP'Q'$ be $y = mx$, then that to the conic will be

$$\lambda \left(\frac{x}{a} + \frac{y}{b} - 1 \right) \left(\frac{x}{a'} + \frac{y}{b'} - 1 \right) - (y - mx)^2 = 0 \dots (1).$$

Hence, putting $y = 0$, we get

$$\lambda \left(\frac{x}{a} - 1 \right) \left(\frac{x}{a'} - 1 \right) - m^2 x^2 = 0,$$

$$\text{or } \frac{1}{x^2} - \left(\frac{1}{a} + \frac{1}{a'} \right) \frac{1}{x} + \left(\frac{1}{aa'} - \frac{m^2}{\lambda} \right) = 0 \dots \dots \dots (2),$$

the roots of which equation are $\frac{1}{\alpha}$, $\frac{1}{\beta}$; whence we see that

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{a} + \frac{1}{a'},$$

or the line OPQ is divided harmonically in A , B , whence O is one of the foci of involution of the system of points P , Q , A , B , so that the chords of contact of *all* conics touching TP , TQ and passing through A , B , cut AB in one of the points O , O' , if O' be the other focus of involution.

Now, in order that (1) may represent a rectangular hyperbola, the sum of the coefficients of x^2 and y^2 must $= 0$; hence

$$\frac{\lambda}{aa'} - m^2 + \frac{\lambda}{bb'} - 1 = 0.$$

But by (2),

$$\frac{1}{\alpha\beta} = \frac{1}{aa'} - \frac{m^2}{\lambda}.$$

Combining these equations, we get

$$m^2 \left(\frac{1}{aa'} + \frac{1}{bb'} \right) = (m^2 + 1) \left(\frac{1}{aa'} - \frac{1}{\alpha\beta} \right),$$

giving two values for m , equal and of opposite signs, so that there can be constructed two pair of rectangular hyperbolæ

whose chords of contact meet in one of the foci of involution, and are equally inclined to AB .

The relation between the four lines Ox , Oy , Op , Oq , in the problem on p. 158, may be expressed thus: Ox and Oy each bisect the lines between Op , Oq parallel to the other.

For let the equations to Op , Oq , referred to Ox , Oy as axes, be $y = mx$, $y = m'x$. Then if $Oa = a$, $Ob = b$, $Oa' = a'$, $Ob' = b'$, the equations to ab , $a'b'$ are

$$\frac{x}{a} + \frac{y}{b} = 1,$$

$$\frac{x}{a'} + \frac{y}{b'} = 1;$$

whence, if (xy) be the point Q of intersection of these lines,

$$\left(\frac{a}{b} - \frac{a'}{b'}\right)y = (a - a'),$$

$$\left(\frac{b}{a} - \frac{b'}{a'}\right)x = (b - b').$$

But Q lies on the line Oq , or $y = mx$,

$$\therefore (a - a') \left(\frac{b}{a} - \frac{b'}{a'}\right) = m(b - b') \left(\frac{a}{b} - \frac{a'}{b'}\right);$$

$$\therefore \frac{1}{a'} - \frac{1}{a} + m \left(\frac{1}{b'} - \frac{1}{b}\right) = 0.$$

The condition that the point of intersection of this line lies on Op or $y = m'x$, is derived from this equation by interchanging a , a' , and writing m' for m ,

$$\frac{1}{a'} - \frac{1}{a} + m' \left(\frac{1}{b'} - \frac{1}{b}\right) = 0.$$

Hence $m' = -m$, which expresses the above relation between Ox , Oy , Op , Oq .

The same thing may be proved geometrically by making any one of the points a , b , a' , or b' , remove to an infinite distance.

The following statical Problems set in 1850 have been omitted.

1. A heavy rod, whose weight is W , rests upon a fulcrum at its middle point, when loaded at one end with a weight W' , the density at any point of the rod at the distance x from a certain point in it varies as $\sin \frac{\pi x}{a}$, a being the length of the rod: find the ratio of W to W' , and determine at which point the density is zero when this ratio is the greatest possible.

Let c be the distance from the centre of the rod of the point where the density is zero, ρ the density at the point $x = \frac{1}{2}a$. The conditions of the problem give

$$\int_0^{\frac{1}{2}a+c} \rho \sin \frac{\pi x}{a} dx + \int_0^{\frac{1}{2}a-c} \rho \sin \frac{\pi x}{a} dx = W,$$

$$\text{or } \frac{\rho a}{\pi} \left\{ 1 - \cos \left(\frac{\pi}{2} + \frac{\pi c}{a} \right) + 1 - \cos \left(\frac{\pi}{2} - \frac{\pi c}{a} \right) \right\} = W;$$

$$\therefore \rho = \frac{\pi}{2} \cdot \frac{W}{a} \dots \dots \dots (1).$$

Also taking moments about the fulcrum which is at the middle point of the rod,

$$\rho \int_c^{\frac{1}{2}a+c} (x-c) \sin \frac{\pi x}{a} dx = \rho \int_0^c (c-x) \sin \frac{\pi x}{a} dx$$

$$+ \rho \int_0^{\frac{1}{2}a-c} (c+x) \sin \frac{\pi x}{a} dx + W' \frac{a}{2},$$

$$\text{or } \rho \int_{\frac{1}{2}a-c}^{\frac{1}{2}a+c} x \sin \frac{\pi x}{a} dx - \rho c \left(\int_0^{\frac{1}{2}a+c} \sin \frac{\pi x}{a} dx + \int_0^{\frac{1}{2}a-c} \sin \frac{\pi x}{a} dx \right) = W' \frac{a}{2}.$$

$$\text{Now } \int x \sin \frac{\pi x}{a} dx = -\frac{ax}{\pi} \cos \frac{\pi x}{a} + \frac{a^2}{\pi^2} \sin \frac{\pi x}{a} + c;$$

$$\therefore \int_{\frac{1}{2}a-c}^{\frac{1}{2}a+c} x \sin \frac{\pi x}{a} dx = \frac{a}{\pi} \sin \frac{\pi c}{a} \left\{ \frac{a}{2} + c + \left(\frac{a}{2} - c \right) \right\}$$

$$= \frac{a^2}{\pi} \sin \frac{\pi c}{a},$$

and $\int_0^{\frac{1}{2}a+c} \sin \frac{\pi x}{a} dx + \int_0^{\frac{1}{2}a-c} \sin \frac{\pi x}{a} dx = \frac{2a}{\pi}$, as shewn in (1);

$$\therefore \rho \frac{a^2}{\pi} \sin \frac{\pi c}{a} - \frac{2\rho ca}{\pi} = W' \frac{a}{2},$$

$$\text{or from (1)} \quad \frac{\pi}{2} \frac{W}{a} \left(\frac{a^2}{\pi} \sin \frac{\pi c}{a} - \frac{2ac}{\pi} \right) = W' \frac{a}{2};$$

$$\therefore \frac{W}{W'} = \frac{a}{a \sin \frac{\pi c}{a} - 2c}.$$

This ratio = ∞ when $c = 0$; it has a maximum value when

$$a \sin \frac{\pi c}{a} - 2c \text{ is a minimum;}$$

or, differentiating with respect to c ,

$$\pi \cos \frac{\pi c}{a} - 2 = 0,$$

$$\cos \frac{\pi c}{a} = \frac{2}{\pi},$$

which determines the value of c .

2. Portions are cut from an ellipsoid by planes which are parallel and equidistant from the centre; if ϖ be the length of a perpendicular from the centre upon either plane, and l, m, n , the cosines of the angles which it makes with the axes, shew that the remainder will rest when placed with a section on a horizontal plane, if

$$\frac{1}{\varpi^2} = \text{or} > \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2},$$

a, b, c , being the axes of the ellipsoid; and express the condition that p such solids, when placed on each other with their sections coincident, and their centres in a line inclined to the vertical, shall not fall over.

The one portion will rest with a section upon a horizontal plane if the vertical line drawn through its centre of gravity,

which is the centre of the ellipsoid, fall within the section; *i.e.* if ϖ be equal or less than the radius vector (r) of the ellipsoid drawn in the same direction, or since

$$\frac{l^2 r^2}{a^2} + \frac{m^2 r^2}{b^2} + \frac{n^2 r^2}{c^2} = 1,$$

$$\text{if } \frac{1}{\varpi^2} = \text{or} > \frac{1}{r^2} = \text{or} > \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}.$$

When there are p such solids placed on each other as above described, the height of the centre of gravity above the plane will be $p\varpi$; and, if ρ' be the distance of the foot of the perpendicular drawn from the centre of gravity on the horizontal plane from the centre of the section, ρ the same distance when there is but one solid, we have $\rho' = p\rho$: the condition that the p solids shall not fall over is, that $p\rho$ shall be equal to or less than the radius vector of the section through the foot of the said perpendicular.

The equation to the cutting plane is

$$lx + my + nz = \varpi \dots\dots\dots (1);$$

and if α, β, γ , be the coordinates of the centre of the section, α, β, γ , are subject to the conditions (see Gregory's *Solid Geometry*, Art. 121)

$$l\alpha + m\beta + n\gamma = \varpi,$$

$$\text{and } \frac{\alpha}{a^2 l} = \frac{\beta}{b^2 m} = \frac{\gamma}{c^2 n} = \frac{\varpi}{a^2 l^2 + b^2 m^2 + c^2 n^2} \dots\dots\dots (2).$$

The coordinates of the foot of the perpendicular on (1) are $l\varpi, m\varpi, n\varpi$; hence the equations to the radius vector of the section through this foot are

$$\frac{x - \alpha}{l\varpi - \alpha} = \frac{y - \beta}{m\varpi - \beta} = \frac{z - \gamma}{n\varpi - \gamma} = \mu r \text{ suppose } \dots\dots\dots (3),$$

where r is the distance of the point (xyz) from ($\alpha\beta\gamma$).

If we substitute these values of x, y, z , in the equation to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

we shall find the length of the radius vector r of the section through the foot of the perpendicular,

$$\frac{\{(l\varpi - \alpha) \mu r + \alpha\}^2}{a^2} + \frac{\{(m\varpi - \beta) \mu r + \beta\}^2}{b^2} + \frac{\{(n\varpi - \gamma) \mu r + \gamma\}^2}{c^2} = 1,$$

the roots of this equation are equal;

$$\therefore \left\{ \frac{(l\varpi - \alpha)^2}{a^2} + \frac{(m\varpi - \beta)^2}{b^2} + \frac{(n\varpi - \gamma)^2}{c^2} \right\} \mu^2 r^2 = 1 - \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} \right) \dots (4).$$

Now, from equations (2),

$$\frac{l\alpha}{a^2} + \frac{m\beta}{b^2} + \frac{n\gamma}{c^2} = \frac{\varpi}{a^2 l^2 + b^2 m^2 + c^2 n^2},$$

$$\text{and } \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = \frac{\varpi^2}{a^2 l^2 + b^2 m^2 + c^2 n^2};$$

and from equations (3),

$$\begin{aligned} \frac{1}{\mu^2} &= (l\varpi - \alpha)^2 + (m\varpi - \beta)^2 + (n\varpi - \gamma)^2 \\ &= \varpi^2 - 2\varpi^2 + \alpha^2 + \beta^2 + \gamma^2 \\ &= \alpha^2 + \beta^2 + \gamma^2 - \varpi^2 \\ &= \rho^2. \end{aligned}$$

Hence equation (4) becomes

$$\left\{ \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} - \frac{\varpi^2}{a^2 l^2 + b^2 m^2 + c^2 n^2} \right\} \frac{r^2}{\rho^2} = 1 - \frac{\varpi^2}{a^2 l^2 + b^2 m^2 + c^2 n^2}.$$

Now the equation of equilibrium is

$$\frac{\rho^2}{r^2} = \text{or} < \frac{1}{p^2};$$

hence the required condition is

$$\begin{aligned} &\left\{ \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right) (a^2 l^2 + b^2 m^2 + c^2 n^2) - 1 \right\} \varpi^2 \\ &= \text{or} < \frac{1}{p^2} (a^2 l^2 + b^2 m^2 + c^2 n^2 - \varpi^2), \end{aligned}$$

$$\begin{aligned} \text{or } &\left\{ p^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right) (a^2 l^2 + b^2 m^2 + c^2 n^2) - (p^2 - 1) \right\} \varpi^2 \\ &= \text{or} < a^2 l^2 + b^2 m^2 + c^2 n^2. \end{aligned}$$

3. If a plane area, bounded by a parabola and its double ordinate, be supported by an axis through the focus and a vertical force acting along the ordinate, find what portion may be cut off by a line through the focus without affecting the vertical force; and the least area for which this is possible.

Let PSQ (fig. 127) be the line cutting off the portion PQR : the centre of gravity of PQR must lie in the vertical through S , or if we draw RV the diameter of PQ , and SG vertical meeting it in G , G must be the centre of gravity of PQR . Hence $GV = \frac{2}{5}RV$. Let $AS = l$, $\angle PSB = \theta$;

$$\therefore \tan \theta = \frac{2l}{SG};$$

$$\therefore GV = SG \cot \theta = 2l \cot^2 \theta.$$

$$\text{Also } SP = \frac{2l}{1 - \cos \theta};$$

$$\begin{aligned} \therefore PV &= SP - SV = SP - SG \operatorname{cosec} \theta \\ &= \left(\frac{1}{1 - \cos \theta} - \frac{\cos \theta}{\sin^2 \theta} \right) 2l \\ &= \frac{2l}{\sin^2 \theta}. \end{aligned}$$

$$\text{But } PV^2 = \frac{4l}{\sin^2 \theta} \cdot RV = \frac{4l}{\sin^2 \theta} \cdot 5l \cot^2 \theta;$$

$$\text{or } \frac{4l^2}{\sin^2 \theta} = 4l \cdot 5l \cot^2 \theta;$$

$$\therefore \cos^2 \theta = \frac{1}{5},$$

which determines the position of the line PSQ .

If the bounding ordinate have its extremity C nearer the vertex than the point P just determined, let $Q'SP'$ be the position of the cutting line, the centre of gravity of $Q'CP'$ must lie in the vertical through S : draw $Q'u$ vertical: let $AB' = a$;

$$\begin{aligned} \therefore \int_0^{a-l} \{2l^{\frac{1}{2}}(x+l)^{\frac{1}{2}} - x \tan \theta\} x dx \\ = \int_0^{\frac{2l}{1+\sec \theta}} \{2l^{\frac{1}{2}}(l-x)^{\frac{1}{2}} + x \tan \theta\} x dx + 2 \int_{\frac{2l}{1+\sec \theta}}^l 2(lx)^{\frac{1}{2}} x dx. \end{aligned}$$

This equation, when reduced, will determine the value of θ in terms of α .

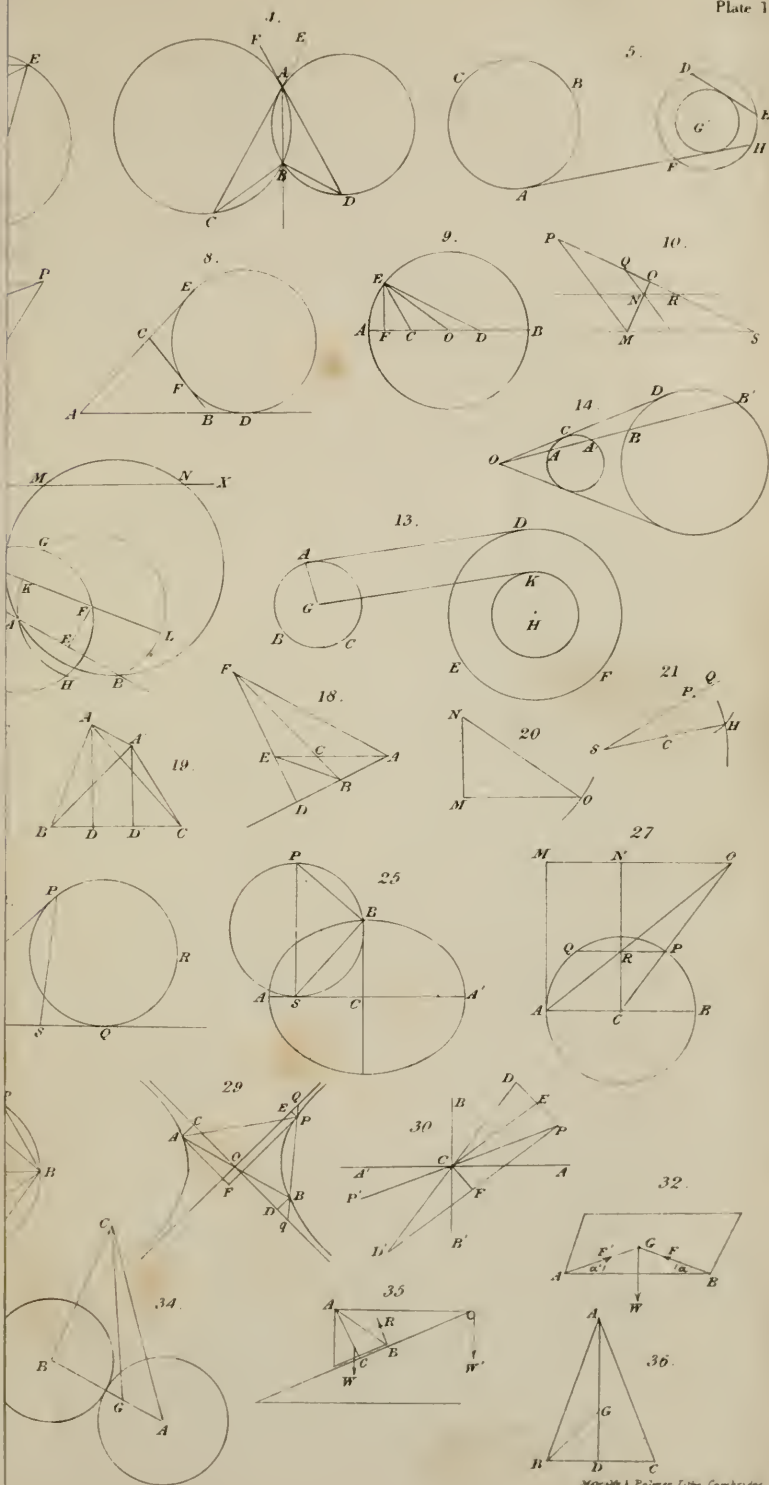
The least area for which this is possible will evidently be such, that the part cut off will be the half, and the cutting line ASB : in this case S is the centre of gravity of the whole area, and $AB = \frac{5}{3}l$.

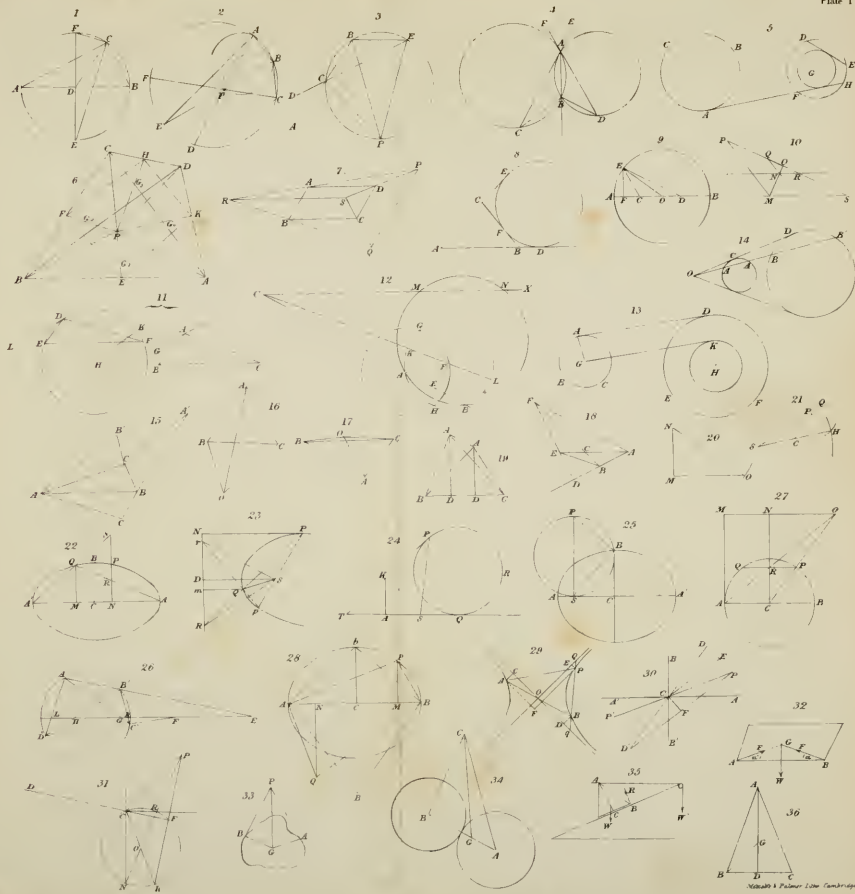
The first part of the Prob. 5, on p. 212, may be proved by referring to the values of the angles contained between any two adjacent sides of a regular polyhedron (see Hall's *Spherical Trigonometry*, Art. 59): it appears that this angle is a sub-multiple of 2π only in the case of the cube.

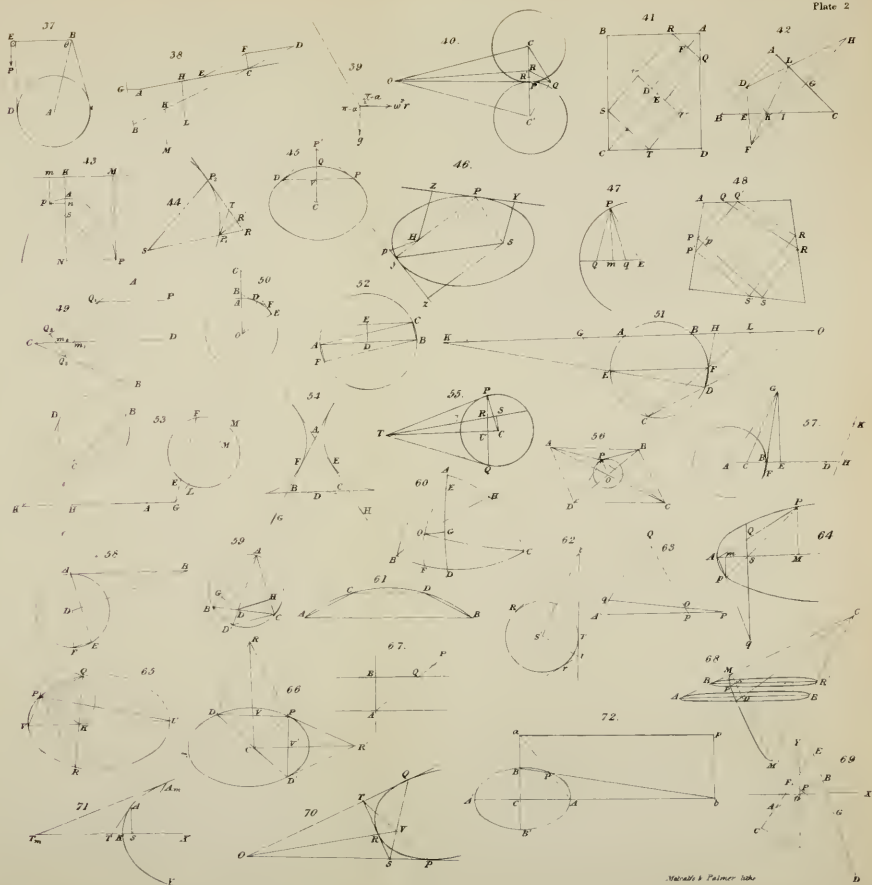
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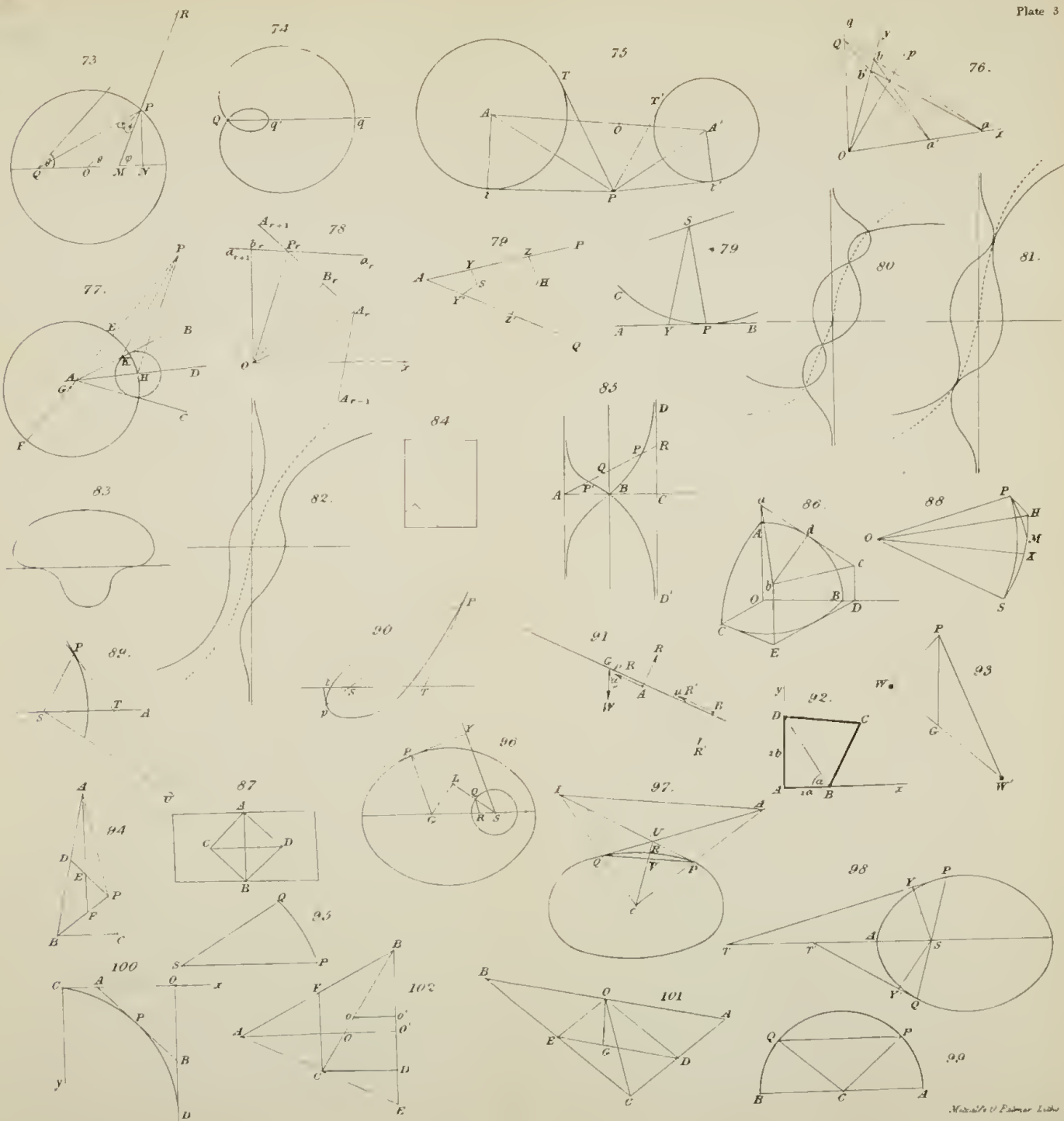
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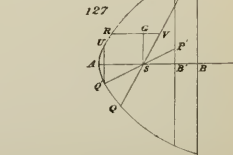
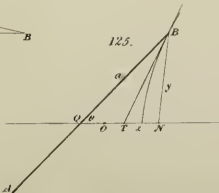
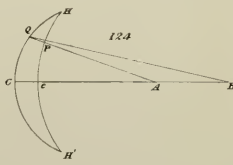
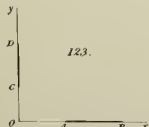
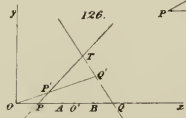
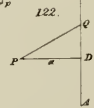
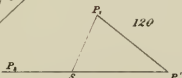
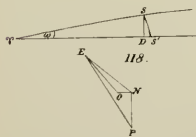
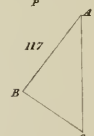
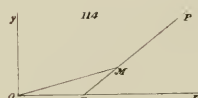
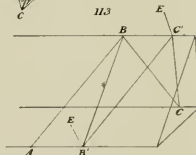
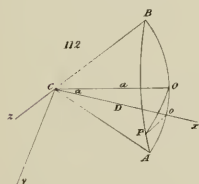
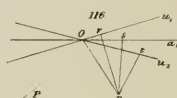
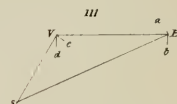
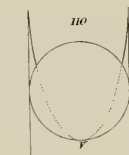
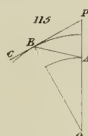
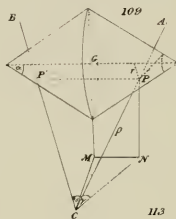
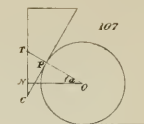
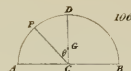
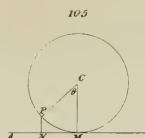
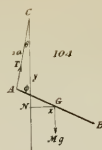
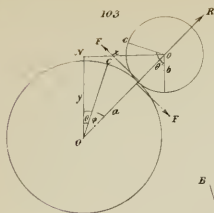
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